On asymptotic behaviour of the increments of sums of i.i.d. random variables from domains of attraction of asymmetric stable laws.

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Let $X, X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables. Put $S_n = X_1 + \ldots + X_n$, $S_0 = 0$.

Let a_n be a nondecreasing sequence of natural numbers.

We will study the asymptotic behaviour of the increments of sums

$$T_n = S_{n+ca_n} - S_n$$

as well as the maximal increments

$$U_n = \max_{0 \le k \le n-a_n} (S_{k+a_n} - S_k).$$

The aim is to describe a normalizing sequence c_n such that

$$\limsup \frac{T_n}{c_n} = 1 \quad a.s.$$

$$\limsup \frac{U_n}{c_n} = 1 \quad a.s.$$

L. Shepp (1964) $T_n = \frac{S_{n+a_n} - S_n}{a_n}$, $a_n \nearrow \infty$, a_n takes positive integer values. $M_t = \mathbf{E}e^{Xt} < \infty$. $T = \limsup T_n$ was determined in terms of the moment generating function of X and the radius of convergence of $\sum x^{a_n}$ (denoted r).
$m(a) = \min M(t)e^{-at}$.
$T = a$ a.s., where $a = a(r)$ is the unique solution of $m(a) = r$.

P. Erdős, A. Rényi (1970) $a_n = [c \log n]$. Theorem 1. Suppose that the moment generating function $M_t = \mathbf{E}e^{Xt}$ exists for $t \in I$, where I is an open interval containing $t = 0$. Let us suppose that $\mathbf{E}X = 0$. Let α be any positive number such that the function $M(t)e^{-\alpha t}$ takes on its minimum in some point in the open interval I and let us put

$$\min_{t \in I} M(t)e^{-\alpha t} = M(\tau)e^{-\alpha\tau} = e^{-1/c}.$$

Then

$$\mathbf{P}(\lim \max_{0 \le k \le n-[c \log n]} \frac{S_{k+[c \log n]} - S_k}{[c \log n]} = \alpha) = 1$$

Theorem 2. The functional dependence between α and $c = c(\alpha)$ determines the distribution of the random variables X_n uniquely.

Practical implements.

1. The longest runs of pure heads.

Theorem 3(special case of Theorem 1). Let $X_1, X_2, ...$ be independent Bernoulli random variables with $\mathbf{P}(X_i = 1) = 0.5 = \mathbf{P}(X_i = -1)$, $S_n = X_1 + \ldots + X_n$. Then for any $c \in (0, 1)$ there exists $n_0 = n_0(c)$ such that

$$\max_{0 \leq k \leq [c \log_2 n]} (S_{k+[c \log_2 n]} - S_k) = [c \log_2 n] \quad a.s.$$

if $n > n_0$.

This theorem guarantees the existence of a run of length $[c \log_2 n]$ when n is large enough.

2. The stochastic geyser problem. $X_1, X_2, \ldots$ - i.i.d.r.v., $F(.)$ is their distribution function. Put $V_n = S_n + R_n$, where R_n is also a r.v. sequence.

Theorem (Bártfai, 1966). Assume that the moment generating function of X_1 exists in a neibourhood of $t = 0$ and $R_n = o(\log n)$. Then, given the values of $\{V_n; n = 1, 2 \ldots\}$, the distribution function $F(.)$ is determined with probability 1, i.e. there exists a r. v. $L(x) = L(V_1, V_2, \ldots, x)$, measurable with a respect of σ-algebra, generated by $V_1, V_2 \ldots$ such that for any given real x, $L(x) = F(x)$.

Proof. For any $c > 0$ we have

$$\lim \max_{0 \le k \le n - [c \log n]} \frac{V_{k+[c \log n]} - V_k}{[c \log n]} =$$

$$\lim \max_{0 \le k \le n - [c \log n]} \frac{S_{k+[c \log n]} - S_k}{[c \log n]} = \alpha(c) \quad a.s.$$

Improvements.

J. Steinebach (1978).

The existence of a moment generating function is a necessary condition. If $M(t) = \mathbf{E}e^{Xt} = \infty$ for all $t > 0$, then

$$\limsup \max_{0 \le k \le n-[c \log n]} \frac{S_{k+[c \log n]} - S_k}{[c \log n]} = \infty \quad a.s.$$

D. Mason.(1989) (The extended version of Erdős-Rényi laws).

$$\max_{0 \le k \le n-a_n} \frac{S_{k+a_n} - S_k}{\gamma(c) a_n} \overset{a.s.}{\to} 1,$$

where $\gamma(c)$ is a constant depending on c and $M(t)$ remains true when $a_n/\log n \to 0$. (Erdős and Rényi had $a_n/\log n \sim c$).

M. Csörgő and J. Steinebach (1981). Theorem. Suppose $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$ and there exists a $t_0 > 0$ such that $M(t) = \mathbf{E}e^{Xt} < \infty$ if $|t| < t_0$. Then for the sums S_n the following holds

$$\lim_{n\to\infty} \max_{0\le k\le n-a_n} \frac{S_{k+a_n} - S_k}{(2a_n \log(n/a_n))^{1/2}} = 1 \quad a.s.,$$

where $\frac{a_n}{(\log n)^2} \to \infty$.
In this case the normalizing sequence depends only on the moment conditions on X.

$$T_n = S_{n+ca_n} - S_n$$

$$U_n = \max_{0 \le k \le n-a_n} (S_{k+a_n} - S_k), \quad \limsup \frac{U_n}{c_n} = 1 \quad a.s.$$

The asymptotic behahior of U_n and T_n strongly depends on the rate of the growth of a_n and the moment conditions on X.
If $a_n = O(\log n)$, the normalizing sequence c_n depends on the distribution of X (Erdős-Rényi laws).
If $a_n/\log n \to \infty$ and $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, the normalising sequence does not depend on the distribution of X and is the same as the one for the Gaussian distribution. In this case $c_n = \sqrt{2a_n(\log(n/a_n) + \log\log n)}$ (Csörgő-Révész laws).
For example: put $a_n = n$, $c_n = (2n \log\log n)^{1/2}$, $U_n = S_n$,

$$\limsup \frac{S_n}{\sqrt{2n \log\log n}} = 1 \quad a.s.$$

Frolov (2000).
It turned out, that these two types of behaviour are particular cases of the universal one. For variables with a finite moment generating function there exists an explicit formula for the normalizing sequence c_n.

H. Lanzinger, U. Stadtmuller.
Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables. Suppose $\mathbf{E}X = 0$, $\mathbf{E}X^2 = \sigma^2$. $\mathbf{E}e^{t|X|^{1/p}} < \infty$ for all t in a neibourhood of 0.

$$t_0 = \sup\{t \geq 0 : \mathbf{E}e^{g(tX)} < \infty\} \in (0, \infty)$$

$$\varphi(c) = \max\{x + y : \frac{x^2}{2c\sigma^2} + (t_0 y)^{\frac{1}{p}} \leq 1, x \geq 0, y \geq 0\}.$$

Theorem.
Under assumptions made above, we have

$$\lim_{n\to\infty} \max_{0\leq j<n} \max_{1\leq k\leq n-j} \frac{S_{j+k} - S_j}{\varphi(\frac{k}{(\log n)^{2p-1}})(\log n)^p} = 1 \quad \text{a.s.}$$

Corollary

$$\limsup_{n\to\infty} \max_{0\le j<n} \frac{S_{j+c(\log n)^{2p-1}} - S_j}{\varphi(c)(\log n)^p} = 1 \quad \text{a.s.}$$

H. Lanzinger (2000).
Theorem.

$$\limsup_{n\to\infty} \frac{S_{n+(\log n)^p} - S_n}{(\log n)^{(p+1)/2}} = \varphi(1) \quad \text{a.s.}$$

Definition. Suppose that X has a distribution R. The distribution R is stable if for every n there exist $c_n > 0$ and γ_n such that $S_n = c_n X + \gamma_n$. $c_n = n^{1/\alpha} c$, $0 < \alpha \leq 2$.
Normal distribution is stable with $\alpha = 2$ and $\gamma_n = 0$.
The distribution function G belongs to the domain of attraction of R if there exist a sequence B_n, $B_n > 0$ and A_n, such that

$$\frac{S_n - A_n}{B_n} \xrightarrow{d} R.$$

There exists a canonical representation of the characteristic function of a stable law.

$$f(t) = \exp(it\gamma - c|t|^{\alpha}(1 - i\frac{t}{|t|}\beta\omega(t, \alpha))),$$

where $\gamma \in R$, $c \geq 0$, $|\beta| \leq 1$, $\omega(t, \alpha) = \tan \pi\alpha/2$ if $\alpha \neq 1$ and $\omega(t, \alpha) = (2/\pi) \log t$, if $\alpha = 1$.

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, $\mathbf{E}X = 0$, $F(x) = \mathbf{P}(X < x)$. Suppose $F(x)$ to be from a domain of attraction of a stable law with index $\alpha \in (1, 2)$ and the characteristic function $\psi(t) = \exp\{-a|t|^\alpha(1 + i\frac{t}{|t|}\tan\frac{\pi}{2}\alpha)\}$, $a = \cos(\pi(2-\alpha)/2)$. Let $B_n = n^{\frac{1}{\alpha}}$.

Define, further

$c_n = (\log n)^{\frac{p+\alpha-1}{\alpha}}, \quad t_0 = \sup\{t \geq 0 : \mathbf{E}e^{t(X^+)^{\frac{\alpha}{p+\alpha-1}}} < \infty\}$,

$$\varphi(c) = \max\{x + y : \frac{(\alpha-1)x^{\frac{\alpha}{\alpha-1}}}{\alpha c^{\frac{1}{\alpha-1}}} + t_0 y^{\frac{\alpha}{p+\alpha-1}} \leq 1, x \geq 0, y \geq 0\}.$$

Theorem. Suppose $t_0 \in (0, \infty)$

Then

$$\limsup_{n\to\infty} \frac{S_{n+ca_n} - S_n}{c_n\varphi(c)} = 1 \quad \text{a.s.}$$

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