New general conditions for the existence of stochastic integrals

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Plan

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- Examples
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- Sharpness of these bounds

Motivation

- We will deal a lot with Riemann-Stieltjes and forward integrals
- These integrals have natural interpretation in financial mathematics
- Riemann-Stieltjes sums appear in many econometric applications (e.g. unit root tests) and their bounds are useful for limit theorems used in tests
- Riemann-Stieltjes integrals are of interest because they represent linear functionals in certain Banach spaces

- Let $X = \{X(t) : t \in [a, b]\}$ be a stochastic process

  ϕ increase on $\mathbb{R}_+$ with $\phi(0) = 0$

  $\| \ \|$ be a norm on random variables

  Define $(\phi, \| \ \|)$-*variation* of X on $[a, b]$ as

$$\sup \sum_{i=0}^{n-1} \phi\big(\|X(t_{i+1}) - X(t_i)\|\big) \ \text{ with } \ a = t_0 < \ldots < t_n = b$$

- Define *Riemann-Stieltjes integral* $\int_a^b U dV$ in $\| \ \|$ as

$$\lim \sum_{i=0}^{n-1} U(s_i)[V(t_{i+1}) - V(t_i)] \ \text{ with } \ s_i \in [t_i, t_{i+1}]$$

  Put $s_i = t_i$ to get a *forward integral*

Young's Theorem (1938).

Let U, V have finite $(\varphi, \|\ \|_u)$- and $(\psi, \|\ \|_v)$-variations, respectively.

Suppose U, V have limits on the right and left in $\|\ \|_u$ and $\|\ \|_v$ as well as have no common discontinuities. If

$$\int_0^1 \frac{\varphi^{-1}(x)\psi^{-1}(x)}{x^2}\, dx < +\infty,$$

then Riemann-Stieltjes integral $\int_a^b U dV$ exists as $\lim$ in any $\|\ \|$ satisfying

$$\|\xi\eta\| \leqslant \|\xi\|_u \cdot \|\eta\|_v.$$

See also Lesniewicz&Orlicz(1973), Dyac'kov(1988), Dyac'kov(1996)

- $\| \ \|_u, \| \ \|_v, \| \ \|$ are L_p, L_q, L_r-norms, and

$$\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{r}, \quad p, q, r \geqslant 1$$

Further on $B_{\mathbb{H}}$ is a fBM with Hurst index $\mathbb{H} > 1/2$

- $\int_a^b f(B_{\mathbb{H}}) dB_{\mathbb{H}}$ exists a.s. and in L_1 for smooth f
- $\int_a^b \mathbf{1}\big(B_{\mathbb{H}}(t) \geqslant c\big) dB_{\mathbb{H}}$ exists in L_1 whenever $c > 0$ or $a > 0$

BUT: easy to see that on $[0,b]$, $b>0$,

$\mathbf{1}\big(B_{\mathbb{H}}(t)\geqslant 0\big)$ has infinite $(\phi,\|\ \|)$-variation for all ϕ and any $\|\ \|$ depending only on the distribution of a random variable.

Does the integral $\int_{[0,1]}\mathbf{1}\big(B_{\mathbb{H}}\geqslant 0\big)dB_{\mathbb{H}}$ exist?

The latter appears in Tanaka formula, see Nualart(2001).

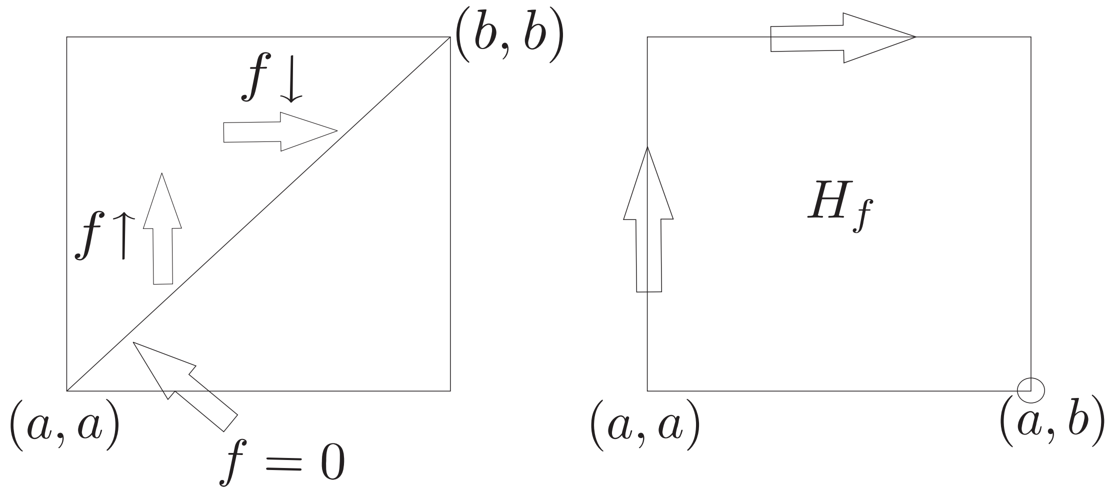
Let

f decreases in x on $[a, y]$ and increases in y on $[x, b]$

f is symmetric and $f(x, x) = 0$

Introduce

$$H_f(a,b) = \iint\limits_{[a,b]^2} \frac{f(x,y)}{(x-y)^2}\,dxdy + \int\limits_a^b \frac{f(a,y)}{y-a}\,dy + \int\limits_a^b \frac{f(x,b)}{b-x}\,dx + f(a,b)$$



Theorem.

Let

$$\|[U(u) - U(x)] \cdot [V(y) - V(v)]\| \leqslant f(x, y), \quad u, v \in [x, y]. \qquad (*)$$

Suppose also

$$H_f(a, b) = \iint\limits_{[a,b]^2} \frac{f(x, y)}{(x - y)^2}\, dxdy + ... < \infty.$$

Then Riemann-Stieltjes integral $\int_a^b U dV$ *exists* (as $\lim$ in $\| \, \|$).

If $(*)$ *holds only for* $u = v$*, then forward integral* $\int_a^b U dV$ *exists.*

- Take p sufficiently close to 1 and

$$f(x,y) = const \cdot 2 \wedge \left\{ \frac{|y-x|^{1/(2p)}}{x^{\mathbb{H}/p}} + \frac{|y-x|^{1/p}}{x^{3\mathbb{H}/p}} \right\} \cdot |y-x|^{\mathbb{H}}$$

to see that there exists $\int_{[0,1]} \mathbf{1}\big(B_{\mathbb{H}}(t) \geqslant 0\big) dB_{\mathbb{H}}$

- or more general

$$\int_{[a,b]} F(B_{\mathbb{H}}) dG(B_{\mathbb{H}})$$

with $F(x) = \int_{(-\infty,x]} f(y) Q(dy)$, $\int_{\mathbb{R}} |f(y)|(1+|y|) Q(dy) < \infty$ and G s.t.

$$\sup \frac{\|\Delta G(B_{\mathbb{H}})\|_p}{\|\Delta B_{\mathbb{H}}\|_p} < \infty \text{ for sufficiently large } p \geqslant 1.$$

Similar integrals are considered by Azmoodeh, Mishura, Valkeila(2009)

NOTE:

$\int_a^b U dV$ will not change if we change the time

$$t \to s = \varphi(t), \quad \varphi \text{ is continuous and increases on } [a, b],$$

i.e.

$$\int\limits_a^b U dV = \int\limits_{\psi(a)}^{\psi(b)} (U \circ \psi) d(V \circ \psi), \;\; \psi = \varphi^{-1}.$$

BUT: $f(x, y)$ will change on $f(\psi(x), \psi(y))$.

Changing the time properly we get

Theorem*.

Let $\|[U(u) - U(x)] \cdot [V(y) - V(v)]\| \leqslant f(x, y),\ u, v \in [x, y],$ *where*

$$f(x, y) = F\big(v_1(y) - v_1(x), ..., v_n(y) - v_n(x)\big)$$

with coordinate-wise nondecreasing v_i *and* F.

Suppose U, V *have limits on the right and left in terms of* $\| \ \|$ *and do not have common discontinuities. If*

$$\int\limits_0^1 \frac{F(x, ..., x)}{x^2} dx < \infty,$$

then $\int_a^b U dV$ *exists.*

Young's Theorem (1938).

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Suppose U, V have limits on the right and left in $\| \, \|_u$ and $\| \, \|_v$ as well as have no common discontinuities. If

$$\int_0^1 \frac{\varphi^{-1}(x)\psi^{-1}(x)}{x^2}\, dx < +\infty,$$

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$$\|\xi\eta\| \leqslant \|\xi\|_u \cdot \|\eta\|_v.$$

To get Young's theorem

$$\text{put } n = 2 \text{ and } F(x, y) = \varphi^{-1}(x)\psi^{-1}(y) \text{ in } \textit{Theorem}^*$$

Indeed,

$$\begin{aligned} \|\Delta U \cdot \Delta V\| &\leqslant \varphi^{-1}\Big(\varphi(\|\Delta U\|_u)\Big) \cdot \psi^{-1}\Big(\psi(\|\Delta V\|_v)\Big) \\ &\leqslant \varphi^{-1}\big(v_1(y) - v_1(x)\big) \cdot \psi^{-1}\big(v_2(y) - v_2(x)\big) \end{aligned}$$

$v_1(z)$ is φ-variation of U on $[a, z]$, $v_2(z)$ is defined similarly for V

*Theorem** covers the case of $\int U dV$ with

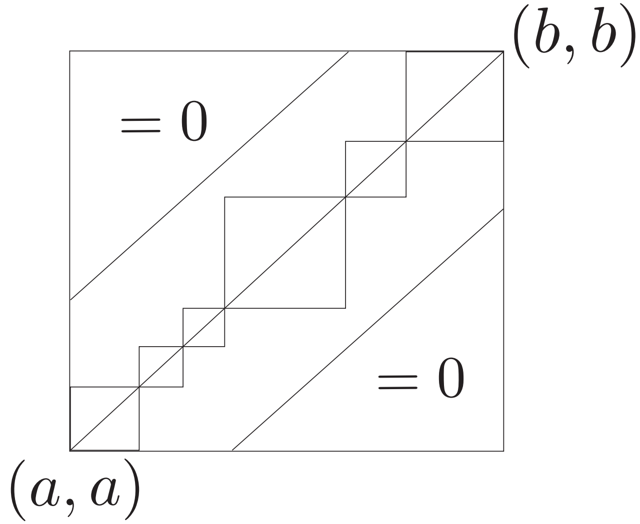
$$U = U_1 ... U_n, \quad V = V_1 ... V_n$$

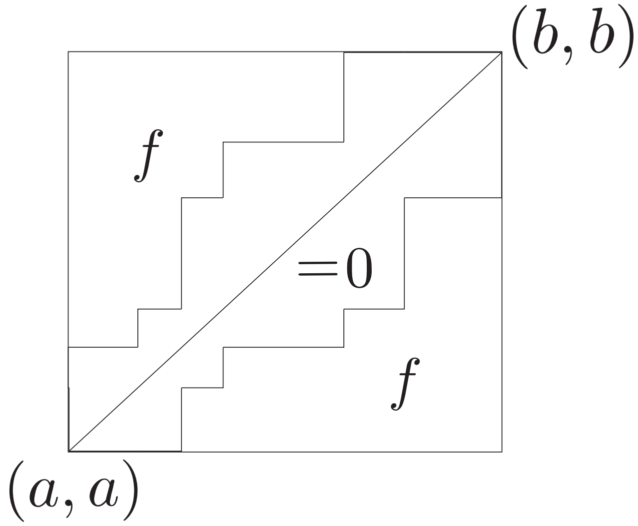
with bounded U_i, V_i of finite φ_i- and ψ_i-variations

Lemma.

$$\sum_{i=0}^{n-1} H_f(t_i, t_{i+1}) \leqslant 9 H_{f\mathbf{1}_{C(d)}}(t_{\min}, t_{\max}),$$

where $C(d) = \{(x, y) \in \mathbb{R}^2 : |x - y| \leqslant 3d\}$, $d = \max \Delta t_i$.





Lemma*.

$$\Big\| \sum_{i=0}^{n-1} [U(t_i) - U(t_0)][V(t_{i+1}) - V(t_i)] \Big\| \leqslant 4H_{f\mathbf{1}_D}(t_0, t_n),$$

here $D = \{(x,y) : [x,y]\,($*or* $[y,x])$ *contains* $\geqslant 2$ *of* $t_i\}$.

We arrive at the main *Theorem* if $H_f(a,b) < \infty$.

BUT what about $H_f(a,b) = \infty$?

Consider U, V s.t.

$$\|\Delta U\|_2 \leqslant const \cdot |\Delta t|^{1/2},\ U(t_0) = 0 \text{ and } V = B \text{ is BM}$$

here $f(x, y) = const \cdot |x - y|$ implying $H_f(a, b) = \infty$

- If $0 = t_0 < ... < t_n = 1$ then *Lemma** reduces to

$$\Big\| \sum_{i=0}^{n} U(t_i)[B(t_{i+1}) - B(t_i)] \Big\|_1 \leqslant C - C \cdot \sum_{i=0}^{n-1} (t_{i+1} - t_i) \ln(t_{i+1} - t_i)$$

with C positive constant

- If $t_i = i/n$, then the upper bound gives $O(\ln n)$

Example 1 (uniform partition of $[0,1]$*)...*

Wiener's construction of BM on $[0,1]$:

$$Z(t) = \zeta_0(\mathbf{1}_{[0,t]}, e_0)_{L_2} + 2\sum_{n\geqslant 1} \zeta_n(\mathbf{1}_{[0,t]}, e_n)_{L_2},$$

here $e_n(x) = e^{2\pi i n x}$, $\zeta_n = (\xi_n + i\eta_n)/\sqrt{2}$ with $\xi_n, \eta_n \sim i.i.d.\ \mathcal{N}(0,1)$.

- *If* $(B, B_0) = (\operatorname{Re} Z, \operatorname{Im} Z)$*, then* B *is BM, and* B_0 *is a Brownian bridge.*

The key property is

$$\Big\| \sum_{i=0}^{n-1} B_0(i/n)[B((i+1)/n) - B(i/n)] \Big\|_1 \geqslant K \ln n, \quad K > 0.$$

This implies that forward integral $\int_{[0,1]} B_0 dB$ doesn't exist, see Lyons(1992), Norvaisa(2008).

Example 2 (arbitrary partition of $[0,1]$*)...*

Consider $U(t) = \int_0^1 \mathcal{H}\mathbf{1}_{[0,t]}(s)dB(s)$, here

$$\mathcal{H}f(t) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{f(s)}{t-s}ds$$

is the Hilbert transform of f. Then

$$\Big\|\sum_{i=0}^{n} U(t_i)[B(t_{i+1}) - B(t_i)]\Big\|_1 \geqslant -\frac{1}{2\pi}\sum_{i=0}^{n-1}(t_{i+1}-t_i)\ln(t_{i+1}-t_i)$$

and the forward integral $\int_{[0,1]} U dB$ doesn't exist.

Property of $\mathcal{H}$:

$$\operatorname{Im} f = \mathcal{H}(\operatorname{Re} f) \text{ on } \mathbb{R} \text{ whenever } f(z) \text{ is holomorphic on } \mathbb{C}$$

- *if $Z(z)$ above were holomorphic, then $B_0 = \mathcal{H}B$ on* $[0,1]$

$$\mathcal{H}B(t),\ B(t) \quad \rightleftharpoons \quad \int_{[0,1]} \mathcal{H}\mathbf{1}_{[0,t]}(s)dB(s),\ B(t)$$

Conclusion

- We give a simple extension of Young's conditions for the existence of stochastic integrals
- This covers some integrals appearing in non-semimartingale models of financial markets
- Bounds appearing in the proofs have its own interest
(eg., they allow to derive some LLN useful in econometrics)
- Particular examples show the sharpness of the bounds and closely relate to Wiener's construction of BM