Basic Arbitrage Theory

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Contents

Björk,T. *Arbitrage Theory in Continuous Time.*
3:rd ed. 2009. Oxford University Press.

1.

Mathematics Recap

Ch. 10-12

Contents

1.1

Conditional Expectation

Conditional Expectation

If $\mathcal{F}$ is a sigma-algebra and X is a random variable which is $\mathcal{F}$-measurable, we write this as $X \in \mathcal{F}$.

If $X \in \mathcal{F}$ and if $\mathcal{G} \subseteq \mathcal{F}$ then we write $E[X|\mathcal{G}]$ for the conditional expectation of X given the information contained in $\mathcal{G}$. Sometimes we use the notation $E_{\mathcal{G}}[X]$.

The following proposition contains everything that we will need to know about conditional expectations within this course.

Main Results

Proposition 1: Assume that $X \in \mathcal{F}$, and that $\mathcal{G} \subseteq \mathcal{F}$. Then the following hold.

- The random variable $E[X|\mathcal{G}]$ is completely determined by the information in $\mathcal{G}$ so we have

$$E[X|\mathcal{G}] \in \mathcal{G}$$

- If we have $Y \in \mathcal{G}$ then Y is completely determined by $\mathcal{G}$ so we have

$$E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$$

  In particular we have

$$E[Y|\mathcal{G}] = Y$$

- If $\mathcal{H} \subseteq \mathcal{G}$ then we have the "law of iterated expectations"

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$$

- In particular we have

$$E[X] = E[E[X|\mathcal{G}]]$$

1.2

Changing Measures

Absolute Continuity

Definition: Given two probability measures P and Q on $\mathcal{F}$ we say that Q is **absolutely continuous** w.r.t. P on $\mathcal{F}$ if, for all $A \in \mathcal{F}$, we have

$$P(A) = 0 \quad \Rightarrow \quad Q(A) = 0$$

We write this as

$$Q << P.$$

If $Q << P$ and $P << Q$ then we say that P and Q are **equivalent** and write

$$Q \sim P$$

Equivalent measures

It is easy to see that P and Q are equivalent if and only if

$$P(A) = 0 \quad \Leftrightarrow \quad Q(A) = 0$$

or, equivalently,

$$P(A) = 1 \quad \Leftrightarrow \quad Q(A) = 1$$

Two equivalent measures thus agree on all certain events and on all impossible events, but can disagree on all other events.

Simple examples:

- All non degenerate Gaussian distributions on R are equivalent.
- If P is Gaussian on R and Q is exponential then $Q << P$ but not the other way around.

Absolute Continuity ct'd

Consider a given probability measure P and a random variable $L \geq 0$ with $E^P[L] = 1$. Now **define** Q by

$$Q(A) = \int_A L dP$$

then it is easy to see that Q is a probability measure and that $Q << P$.

A natural question is now if **all** measures $Q << P$ are obtained in this way. The answer is yes, and the precise (quite deep) result is as follows.

The Radon Nikodym Theorem

Consider two probability measures P and Q on $(\Omega, \mathcal{F})$, and assume that $Q << P$ on $\mathcal{F}$. Then there exists a unique random variable L with the following properties

1. $Q(A) = \int_A L dP, \quad \forall A \in \mathcal{F}$
2. $L \geq 0, \quad P - a.s.$
3. $E^P[L] = 1,$
4. $L \in \mathcal{F}$

The random variable L is denoted as

$$L = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}$$

and it is called the **Radon-Nikodym derivative** of Q w.r.t. P on $\mathcal{F}$, or the **likelihood ratio** between Q and P on $\mathcal{F}$.

A simple example

The Radon-Nikodym derivative L is intuitively the local scale factor between P and Q. If the sample space Ω is finite so $\Omega = \{\omega_1, \ldots, \omega_n\}$ then P is determined by the probabilities $p_1, \ldots, p_n$ where

$$p_i = P(\omega_i) \quad i = 1, \ldots, n$$

Now consider a measure Q with probabilities

$$q_i = Q(\omega_i) \quad i = 1, \ldots, n$$

If $Q << P$ this simply says that

$$p_i = 0 \quad \Rightarrow \quad q_i = 0$$

and it is easy to see that the Radon-Nikodym derivative $L = dQ/dP$ is given by

$$L(\omega_i) = \frac{q_i}{p_i} \quad i = 1, \ldots, n$$

If $p_i = 0$ then we also have $q_i = 0$ and we can define the ratio q_i/p_i arbitrarily.

If $p_1, \dots, p_n$ as well as $q_1, \dots, q_n$ are all positive, then we see that $Q \sim P$ and in fact

$$\frac{dP}{dQ} = \frac{1}{L} = \left(\frac{dQ}{dP}\right)^{-1}$$

as could be expected.

Computing expected values

A main use of Radon-Nikodym derivatives is for the computation of expected values.

Suppose therefore that $Q << P$ on $\mathcal{F}$ and that X is a random variable with $X \in \mathcal{F}$. With $L = dQ/dP$ on $\mathcal{F}$ then have the following result.

Proposition 3: With notation as above we have

$$E^Q[X] = E^P[L \cdot X]$$

The Abstract Bayes' Formula

We can also use Radon-Nikodym derivatives in order to compute conditional expectations. The result, known as the abstract **Bayes' Formula**, is as follows.

Theorem 4: Consider two measures P and Q with $Q << P$ on $\mathcal{F}$ and with

$$L^{\mathcal{F}} = \frac{dQ}{dP} \quad \text{on } \mathcal{F}$$

Assume that $\mathcal{G} \subseteq \mathcal{F}$ and let X be a random variable with $X \in \mathcal{F}$. Then the following holds

$$E^{Q}\left[X \middle| \mathcal{G}\right] = \frac{E^{P}\left[L^{\mathcal{F}} X \middle| \mathcal{G}\right]}{E^{P}\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]}$$

Dependence of the σ-algebra

Suppose that we have $Q << P$ on $\mathcal{F}$ with

$$L^{\mathcal{F}} = \frac{dQ}{dP} \quad \text{on } \mathcal{F}$$

Now consider smaller σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. Our problem is to find the R-N derivative

$$L^{\mathcal{G}} = \frac{dQ}{dP} \quad \text{on } \mathcal{G}$$

We recall that $L^{\mathcal{G}}$ is characterized by the following properties

1. $Q(A) = E^P\left[L^{\mathcal{G}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$
2. $L^{\mathcal{G}} \geq 0$
3. $E^P\left[L^{\mathcal{G}}\right] = 1$
4. $L^{\mathcal{G}} \in \mathcal{G}$

A natural guess would perhaps be that $L^{\mathcal{G}} = L^{\mathcal{F}}$, so let us check if $L^{\mathcal{F}}$ satisfies points 1-4 above.

By assumption we have

$$Q(A) = E^P\left[L^{\mathcal{F}} \cdot I_A\right] \quad \forall A \in \mathcal{F}$$

Since $\mathcal{G} \subseteq \mathcal{F}$ we then have

$$Q(A) = E^P\left[L^{\mathcal{F}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$$

so point 1 above is certainly satisfied by $L^{\mathcal{F}}$. It is also clear that $L^{\mathcal{F}}$ satisfies points 2 and 3. It thus seems that $L^{\mathcal{F}}$ is also a natural candidate for the R-N derivative $L^{\mathcal{G}}$, but the problem is that we do not in general have $L^{\mathcal{F}} \in \mathcal{G}$.

This problem can, however, be fixed. By iterated expectations we have, for all $A \in \mathcal{G}$,

$$E^P\left[L^{\mathcal{F}} \cdot I_A\right] = E^P\left[E^P\left[\left. L^{\mathcal{F}} \cdot I_A \right| \mathcal{G}\right]\right]$$

Since $A \in \mathcal{G}$ we have

$$E^P\left[L^{\mathcal{F}} \cdot I_A \middle| \mathcal{G}\right] = E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right] I_A$$

Let us now define $L^{\mathcal{G}}$ by

$$L^{\mathcal{G}} = E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]$$

We then obviously have $L^{\mathcal{G}} \in \mathcal{G}$ and

$$Q(A) = E^P\left[L^{\mathcal{G}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$$

It is easy to see that also points 2-3 are satisfied so we have proved the following result.

A formula for $L^{\mathcal{G}}$

Proposition 5: If $Q << P$ on $\mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$ then, with notation as above, we have

$$L^{\mathcal{G}} = E^{P}\left[L^{\mathcal{F}} \middle| \mathcal{G} \right]$$

The likelihood process on a filtered space

We now consider the case when we have a probability measure P on some space Ω and that instead of just one σ-algebra $\mathcal{F}$ we have a **filtration**, i.e. an increasing family of σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$.

The interpretation is as usual that $\mathcal{F}_t$ is the information available to us at time t, and that we have $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

Now assume that we also have another measure Q, and that for some fixed T, we have $Q << P$ on $\mathcal{F}_T$. We define the random variable L_T by

$$L_T = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T$$

Since $Q << P$ on $\mathcal{F}_T$ we also have $Q << P$ on $\mathcal{F}_t$ for all $t \leq T$ and we define

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \leq t \leq T$$

For every t we have $L_t \in \mathcal{F}_t$, so L is an adapted process, known as the **likelihood process**.

The L process is a P martingale

We recall that

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \leq t \leq T$$

Since $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$ we can use Proposition 5 and deduce that

$$L_s = E^P \left[L_t \middle| \mathcal{F}_s \right] \quad s \leq t \leq T$$

and we have thus proved the following result.

Proposition: Given the assumptions above, the likelihood process L is a P-martingale.

Where are we heading?

We are now going to perform measure transformations on Wiener spaces, where P will correspond to the objective measure and Q will be the risk neutral measure.

For this we need define the proper likelihood process L and, since L is a P-martingale, we have the following natural questions.

- What does a martingale look like in a Wiener driven framework?
- Suppose that we have a P-Wiener process W and then change measure from P to Q. What are the properties of W under the new measure Q?

These questions are handled by the Martingale Representation Theorem, and the Girsanov Theorem respectively.

1.3

The Martingale Representation Theorem

Intuition

Suppose that we have a Wiener process W under the measure P. We recall that if h is adapted (and integrable enough) and if the process X is defined by

$$X_t = x_0 + \int_0^t h_s dW_s$$

then X is a a martingale. We now have the following natural question:

Question: Assume that X is an arbitrary martingale. Does it then follow that X has the form

$$X_t = x_0 + \int_0^t h_s dW_s$$

for some adapted process h?

In other words: Are **all** martingales stochastic integrals w.r.t. W?

Answer

It is immediately clear that all martingales can **not** be written as stochastic integrals w.r.t. W. Consider for example the process X defined by

$$X_t = \begin{cases} 0 & \text{for} \quad 0 \leq t < 1 \\ Z & \text{for} \quad t \geq 1 \end{cases}$$

where Z is an random variable, independent of W, with $E[Z] = 0$.

X is then a martingale (why?) but it is clear (how?) that it cannot be written as

$$X_t = x_0 + \int_0^t h_s dW_s$$

for any process h.

Intuition

The intuitive reason why we cannot write

$$X_t = x_0 + \int_0^t h_s dW_s$$

in the example above is of course that the random variable Z "has nothing to do with" the Wiener process W. In order to exclude examples like this, we thus need an assumption which guarantees that our probability space only contains the Wiener process W and nothing else.

This idea is formalized by assuming that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ **is the one generated by the Wiener process** W.

The Martingale Representation Theorem

Theorem. Let W be a P-Wiener process and assume that the filtation is the **internal** one i.e.

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\left\{W_s;\ 0 \leq s \leq t\right\}$$

Then, for every $(P, \mathcal{F}_t)$-martingale X, there exists a real number x and an adapted process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

i.e.

$$dX_t = h_t dW_t.$$

Proof: Hard. This is very deep result.

Note

For a given martingale X, the Representation Theorem above guarantees the existence of a process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

The Theorem does **not**, however, tell us how to find or construct the process h.

1.4

The Girsanov Theorem

Setup

Let W be a P-Wiener process and fix a time horizon T. Suppose that we want to change measure from P to Q on $\mathcal{F}_T$. For this we need a P-martingale L with $L_0 = 1$ to use as a likelihood process, and a natural way of constructing this is to choose a process g and then define L by

$$\begin{cases} dL_t &= g_t dW_t \\ L_0 &= 1 \end{cases}$$

This definition does not guarantee that $L \geq 0$, so we make a small adjustment. We choose a process φ and define L by

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

The process L will again be a martingale and we easily obtain

$$L_t = e^{\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds}$$

Thus we are guaranteed that $L \geq 0$. We now change measure form P to Q by setting

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \leq t \leq T$$

The main problem is to find out what the properties of W are, under the new measure Q. This problem is resolved by the **Girsanov Theorem**.

The Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem: Choose an adapted process φ, and define the process L by

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

Assume that $E^P[L_T] = 1$, and define a new mesure Q on $\mathcal{F}_T$ by

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \le t \le T$$

Then $Q << P$ and the process W^Q, defined by

$$W_t^Q = W_t - \int_0^t \varphi_s ds$$

is Q-Wiener. We can also write this as

$$dW_t = \varphi_t dt + dW_t^Q$$

Changing the drift in an SDE

The single most common use of the Girsanov Theorem is as follows.

Suppose that we have a process X with P dynamics

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where μ and σ are adapted and W is P-Wiener.

We now do a Girsanov Transformation as above, and the question is what the Q-dynamics look like.

From the Girsanov Theorem we have

$$dW_t = \varphi_t dt + dW_t^Q$$

and substituting this into the P-dynamics we obtain the Q dynamics as

$$dX_t = \{\mu_t + \sigma_t \varphi_t\} dt + \sigma_t dW_t^Q$$

Moral: The drift changes but the diffusion is unaffected.

The Converse of the Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem. Assume that:

- $Q << P$ on $\mathcal{F}_T$, with likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t \; 0, \leq t \leq T$$

- The filtation is the **internal** one .i.e.

$$\mathcal{F}_t = \sigma \left\{ W_s; \; 0 \leq s \leq t \right\}$$

Then there exists a process φ such that

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

2.

The Martingale Approach

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Financial Markets

Price Processes:

$$S_t = \left[S_t^0, ..., S_t^N\right]$$

Example: (Black-Scholes, $S^0 := B,\ S^1 := S$)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Portfolio:

$$h_t = \left[h_t^0, ..., h_t^N\right]$$

$h_t^i =$ number of units of asset i at time t.

Value Process:

$$V_t^h = \sum_{i=0}^{N} h_t^i S_t^i = h_t S_t$$

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i$$

Major insight:
If the price process S is a **martingale**, and if h is **self-financing**, then V is a **martingale**.

NB! This simple observation is in fact the basis of the following theory.

Arbitrage

The portfolio u is an **arbitrage** portfolio if

- The portfolio strategy is self financing.
- $V_0 = 0$.
- $V_T \geq 0, \; P - a.s.$
- $P\left(V_T > 0\right) > 0$

Main Question: When is the market free of arbitrage?

First Attempt

Proposition: If $S_t^0, \cdots, S_t^N$ are P-martingales, then the market is free of arbitrage.

Proof:
Assume that V is an arbitrage strategy. Since

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i,$$

V is a P-martingale, so

$$V_0 = E^P\left[V_T\right] > 0.$$

This contradicts $V_0 = 0$.

True, but useless.

Example: (Black-Scholes)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

(We would have to assume that $\alpha = r = 0$)

We now try to improve on this result.

Choose S_0 as numeraire

Definition:
The **normalized price vector** Z is given by

$$Z_t = \frac{S_t}{S_t^0} = \left[1, Z_t^1, ..., Z_t^N\right]$$

The **normalized value process** V^Z is given by

$$V_t^Z = \sum_0^N h_t^i Z_t^i.$$

Idea:
The arbitrage and self financing concepts should be independent of the accounting unit.

Invariance of numeraire

Proposition: One can show (see the book) that

- S-arbitrage $\iff$ Z-arbitrage.
- S-self-financing $\iff$ Z-self-financing.

Insight:

- If h self-financing then

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

- Thus, if the **normalized** price process Z is a P-martingale, then V^Z is a martingale.

Second Attempt

Proposition: If $Z_t^0, \cdots, Z_t^N$ are P-martingales, then the market is free of arbitrage.

True, but still fairly useless.

Example: (Black-Scholes)

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

$$
\begin{aligned}
dZ_t^1 &= (\alpha - r) Z_t^1 dt + \sigma Z_t^1 dW_t, \\
dZ_t^0 &= 0 dt.
\end{aligned}
$$

We would have to assume "risk-neutrality", i.e. that $\alpha = r$.

Arbitrage

Recall that h is an arbitrage if

- h is self financing
- $V_0 = 0$.
- $V_T \geq 0, \; P - a.s.$
- $P\left(V_T > 0\right) > 0$

Major insight

This concept is invariant under an **equivalent change of measure!**

Martingale Measures

Definition: A probability measure Q is called an **equivalent martingale measure** (EMM) if and only if it has the following properties.

- Q and P are equivalent, i.e.

$$Q \sim P$$

- The normalized price processes

$$Z_t^i = \frac{S_t^i}{S_t^0}, \quad i = 0, \dots, N$$

  are **Q-martingales**.

Wan now state the main result of arbitrage theory.

First Fundamental Theorem

Theorem: The market is arbitrage free

iff

there exists an equivalent martingale measure.

Note:

- The martingale measure will depend on your choice of numeraire.
- The martingale measure (if it exists) is not necessarily unique.

Comments

- It is very easy to prove that existence of EMM imples no arbitrage (see below).
- The other imnplication is technically very hard.
- For discrete time and finite sample space Ω the hard part follows easily from the separation theorem for convex sets.
- For discrete time and more general sample space we need the Hahn-Banach Theorem.
- For continuous time the proof becomes technically very hard, mainly due to topological problems. See the textbook.

Proof that EMM implies no arbitrage

This is basically done above. Assume that there exists an EMM denoted by Q. Assume that $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$. Then, since $P \sim Q$ we also have $Q(V_T \geq 0) = 1$ and $Q(V_T > 0) > 0$.

Recall:

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

Q is a martingale measure

$$\Downarrow$$

V^Z is a Q-martingale

$$\Downarrow$$

$$V_0 = V_0^Z = E^Q\left[V_T^Z\right] > 0$$

$$\Downarrow$$

No arbitrage

Choice of Numeraire

The **numeraire** price S_t^0 can be chosen arbitrarily. The most common choice is however that we choose S^0 as the **bank account**, i.e.

$$S_t^0 = B_t$$

where

$$dB_t = r_t B_t dt$$

Here r is the (possibly stochastic) short rate and we have

$$B_t = e^{\int_0^t r_s ds}$$

Example: The Black-Scholes Model

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

Look for martingale measure. We set $Z = S/B$.

$$
dZ_t = Z_t(\alpha - r)dt + Z_t \sigma dW_t,
$$

Girsanov transformation on $[0, T]$:

$$
\begin{cases}
dL_t &= L_t \varphi_t dW_t, \\
L_0 &= 1.
\end{cases}
$$

$$
dQ = L_T dP, \quad \text{on } \mathcal{F}_T
$$

Girsanov:

$$
dW_t = \varphi_t dt + dW_t^Q,
$$

where W^Q is a Q-Wiener process.

The Q-dynamics for Z are given by

$$dZ_t = Z_t \left[\alpha - r + \sigma\varphi_t\right] dt + Z_t \sigma dW_t^Q.$$

Unique martingale measure Q, with Girsanov kernel given by

$$\varphi_t = \frac{r - \alpha}{\sigma}.$$

Q-dynamics of S:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Conclusion: The Black-Scholes model is free of arbitrage.

Pricing

We consider a market $B_t, S_t^1, \ldots, S_t^N$.

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent T-claim.

Problem: How do we find an arbitrage free price process $\Pi_t\left[X\right]$ for X?

Solution

The extended market

$$B_t, S_t^1, \ldots, S_t^N, \Pi_t\left[X\right]$$

must be arbitrage free, so there must exist a martingale measure Q for $(B_t, S_t, \Pi_t\left[X\right])$. In particular

$$\frac{\Pi_t\left[X\right]}{B_t}$$

must be a Q-martingale, i.e.

$$\frac{\Pi_t\left[X\right]}{B_t} = E^Q\left[\left.\frac{\Pi_T\left[X\right]}{B_T}\right| \mathcal{F}_t\right]$$

Since we obviously (why?) have

$$\Pi_T\left[X\right] = X$$

we have proved the main pricing formula.

Risk Neutral Valuation

Theorem: For a T-claim X, the arbitrage free price is given by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$

Risk Neutral Valuation

Theorem: For a T-claim X, and the numearire S^0 the arbitrage free price is given by the formula

$$\Pi_t[X] = S_t^0 E^0 \left[\left. \frac{X}{S_T^0} \right| \mathcal{F}_t \right]$$

where E^0 denotes expectation w.r.t. the martingale measure Q^0 associated with the numeraire S^0.

Example: The Black-Scholes Model

Q-dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Simple claim:

$$X = \Phi(S_T),$$

$$\Pi_t [X] = e^{-r(T-t)} E^Q [\Phi(S_T) | \mathcal{F}_t]$$

Kolmogorov $\Rightarrow$

$$\Pi_t [X] = F(t, S_t)$$

where $F(t, s)$ solves the Black-Scholes equation:

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF = 0, \\ F(T, s) = \Phi(s). \end{cases}$$

Problem

Recall the valuation formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

What if there are several different martingale measures Q?

This is connected with the **completeness** of the market.

Hedging

Def: A portfolio is a **hedge** against X ("replicates X") if

- h is self financing
- $V_T = X, \quad P - a.s.$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a natural way of pricing X is

$$\Pi_t\left[X\right] = V_t^h$$

When can we hedge?

Second Fundamental Theorem

The second most important result in arbitrage theory is the following.

Theorem:

The market is complete

iff

the martingale measure Q is unique.

Proof: It is obvious (why?) that if the market is complete, then Q must be unique. The other implication is very hard to prove. It basically relies on duality arguments from functional analysis.

Black-Scholes Model

Q-dynamics

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dW_t^Q, \\
dZ_t &= Z_t \sigma dW_t^Q
\end{aligned}
$$

$$M_t = E^Q\left[e^{-rT} X \middle| \mathcal{F}_t \right],$$

Representation theorem for Wiener processes

$\Downarrow$

there exists g such that

$$M_t = M(0) + \int_0^t g_s dW_s^Q.$$

Thus

$$M_t = M_0 + \int_0^t h_s^1 dZ_s,$$

with $h_t^1 = \frac{g_t}{\sigma Z_t}$.

Result:
X can be replicated using the portfolio defined by

$$\begin{aligned} h_t^1 &= g_t/\sigma Z_t, \\ h_t^B &= M_t - h_t^1 Z_t. \end{aligned}$$

Moral: The Black Scholes model is complete.

Special Case: Simple Claims

Assume X is of the form $X = \Phi(S_T)$

$$M_t = E^Q \left[e^{-rT} \Phi(S_T) \middle| \mathcal{F}_t \right],$$

Kolmogorov backward equation $\Rightarrow M_t = f(t, S_t)$

$$\begin{cases} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} & = & 0, \\ f(T, s) & = & e^{-rT}\Phi(s). \end{cases}$$

Itô $\Rightarrow$

$$dM_t = \sigma S_t \frac{\partial f}{\partial s} dW_t^Q,$$

so

$$g_t = \sigma S_t \cdot \frac{\partial f}{\partial s},$$

Replicating portfolio h:

$$\begin{aligned} h_t^B &= f - S_t \frac{\partial f}{\partial s}, \\ h_t^1 &= B_t \frac{\partial f}{\partial s}. \end{aligned}$$

Interpretation: $f(t, S_t) = V_t^Z$.

Define $F(t,s)$ by

$$F(t,s) = e^{rt} f(t,s)$$

so $F(t,S_t) = V_t$. Then

$$\left\{ \begin{array}{ccc} h_t^B & = & \frac{F(t,S_t) - S_t \frac{\partial F}{\partial s}(t,S_t)}{B_t}, \\ h_t^1 & = & \frac{\partial F}{\partial s}(t,S_t) \end{array} \right.$$

where F solves the **Black-Scholes equation**

$$\left\{ \begin{array}{rcl} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF & = & 0, \\ F(T,s) & = & \Phi(s). \end{array} \right.$$

Main Results

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q

- The market is complete $\Leftrightarrow$ Q is unique.

- Every X must be priced by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

  for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X.

- For a hedgeable claim X, all choices of Q will produce the same price for X:

$$\Pi_t[X] = V_t = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

Completeness vs No Arbitrage
Rule of Thumb

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$R = \text{number of random sources}$$
$$N = \text{number of risky assets}$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim.

Rule of thumb

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \leq R$$

- The market is complete if and only if

$$N \geq R$$

Example:
The Black-Scholes model.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

For B-S we have $N = R = 1$. Thus the Black-Scholes model is arbitrage free and complete.

Stochastic Discount Factors

Given a model under P. For every EMM Q we define the corresponding **Stochastic Discount Factor**, or **SDF**, by

$$D_t = e^{-\int_0^t r_s ds} L_t,$$

where

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

There is thus a one-to-one correspondence between EMMs and SDFs.

The risk neutral valuation formula for a T-claim X can now be expressed under P instead of under Q.

Proposition: With notation as above we have

$$\Pi_t [X] = \frac{1}{D_t} E^P [D_T X | \mathcal{F}_t]$$

Proof: Bayes' formula.

Martingale Property of $S \cdot D$

Proposition: If S is an arbitrary price process, then the process

$$S_t D_t$$

is a P-martingale.

Proof: Bayes' formula.

3.

Change of Numeraire

Ch. 26

General change of numeraire.

Idea: Use a fixed asset price process S_t as numeraire. Define the measure Q^S by the requirement that

$$\frac{\Pi(t)}{S_t}$$

is a Q^S-martingale for every arbitrage free price process $\Pi(t)$.

We assume that we know the risk neutral martingale measure Q, with B as the numeraire.

Constructing $\mathbf{Q}^S$

Fix a T-claim X. From general theory:

$$\Pi_0 [X] = E^Q \left[\frac{X}{B_T} \right]$$

Assume that Q^S exists and denote

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

Then

$$\begin{aligned} \frac{\Pi_0 [X]}{S_0} &= E^S \left[\frac{\Pi_T [X]}{S_T} \right] = E^S \left[\frac{X}{S_T} \right] \\ &= E^Q \left[L_T \frac{X}{S_T} \right] \end{aligned}$$

Thus we have

$$\Pi_0 [X] = E^Q \left[L_T \frac{X \cdot S_0}{S_T} \right],$$

For all $X \in \mathcal{F}_T$ we thus have

$$E^Q\left[\frac{X}{B_T}\right] = E^Q\left[L_T\frac{X \cdot S_0}{S_T}\right]$$

Natural candidate:

$$L_t = \frac{dQ_t^S}{dQ_t} = \frac{S_t}{S_0 B_t}$$

Proposition:

$$\begin{array}{c}\Pi(t)/B_t \text{ is a } Q\text{-martingale.} \\ \Downarrow \\ \Pi(t)/S_t \text{ is a } Q^S\text{-martingale.}\end{array}$$

Proof.

$$E^S\left[\left.\frac{\Pi(t)}{S_t}\right|\mathcal{F}_s\right]=\frac{E^Q\left[\left.L_t\frac{\Pi(t)}{S_t}\right|\mathcal{F}_s\right]}{L_s}$$

$$= \frac{E^Q\left[\left.\frac{\Pi(t)}{B_t S_0}\right|\mathcal{F}_s\right]}{L_s}=\frac{\Pi(s)}{B(s)S_0L_s}$$

$$= \frac{\Pi(s)}{S(s)}. \blacksquare$$

Result

$$\Pi_t[X] = S_t E^S\left[\left.\frac{X}{S_t}\right| \mathcal{F}_t\right]$$

We can observe S_t directly on the market.

Example: $X = S_t \cdot Y$

$$\Pi_t[X] = S_t E^S\left[Y \middle| \mathcal{F}_t\right]$$

Several underlying

$$X = \Phi\left[S_0(T), S_1(T)\right]$$

Assume Φ is linearly homogeous. Transform to Q^0.

$$\begin{aligned}\Pi_t\left[X\right] &= S_0(t)E^0\left[\left.\frac{\Phi\left[S_0(T), S_1(T)\right]}{S_0(T)}\right| \mathcal{F}_t\right] \\ &= S_0(t)E^0\left[\left.\varphi\left(Z_T\right)\right| \mathcal{F}_t\right]\end{aligned}$$

$$\varphi\left(z\right) = \Phi\left[1, z\right], \quad Z_t = \frac{S_1(t)}{S_0(t)}$$

Exchange option

$$X = \max\left[S_1(T) - S_0(T),\ 0\right]$$

$$\Pi_t\left[X\right] = S_0(t) E^0\left[\max\left[Z(T) - 1,\ 0\right] \middle| \mathcal{F}_t\right]$$

European Call on Z with strike price K. Zero interest rate.

Piece of cake!

Identifying the Girsanov Transformation

Assume Q-dynamics of S known as

$$dS_t = r_t S_t dt + S_t v_t dW_t$$

$$L_t = \frac{S_t}{S_0 B_t}$$

From this we immediately have

$$dL_t = L_t v_t dW_t.$$

and we can summarize.

Theorem: The Girsanov kernel is given by the numeraire volatility v_t, i.e.

$$dL_t = L_t v_t dW_t.$$