Point Processes

KTH, February 2011

All notation should be clearly defined. Arguments should be complete and careful. Send your solutions in pdf-form to tomas.bjork@hhs.se no later than March 12.

1. Consider two processes X and Y. We assume that, under a given probability measure P, the processes X and Y are both Poisson processes with constant P-intensities λ and μ respectively. We also assume that X and Y are P-independent. The object of this exercise is to prove that under these assumptions, X and Y have $P-a.s.$ (i.e. with probability one) no common jumps. To show this we choose an arbitrary point in time T, and it is enough to show that we have

$$E\left[\sum_{0\leq t\leq T} \Delta X_t \Delta Y_t\right] = 0. \tag{1}$$

   Your job is to prove the relation (1) and to do this you may proceed as follows.

   (a) Use the independence to calculate $E\left[X_T Y_T\right]$.

   (b) Use stochastic calculus to obtain another formula for the expectation $E\left[X_T Y_T\right]$.

   (c) Compare the new formula with the old one to conclude that (1) holds.

   ....(5p)

2. This is a partial converse of the previous problem. Assume that, under a given probability measure P, the processes X and Y are both Poisson processes with constant P-intensities λ and μ respectively. We assume furthermore that X and Y have no common jumps. The question is whether this implies that X and Y are independent. Investigate this (at least partly) by determining whether the random variables $X_t - X_s$ and $Y_t - Y_s$ are independent for all $s < t$.
   ....(5p)

3. Consider the SDE

$$\begin{aligned} dX_t &= aX_t dt + \beta dN_t, \\ X_0 &= x_0, \end{aligned}$$

where a, β, and x_0 are known constants, and N is a counting process.

(a) Prove that the solution of the SDE can be written

$$X_t = e^{at} x_0 + \beta \int_0^t e^{a(t-s)} dN_s.$$

.... (3p)

(b) Now assume that N is Poisson with constant intensity λ. Compute $E[X_t]$.

....(2p)

4. Assume that the counting process N has the $\mathcal{F}_t^N$ predictable intensity

$$\lambda_t = \alpha g(N_{t-})$$

where g is a non negative known deterministic function, and α is an unknown parameter. Show that the ML estimate of α is given by

$$\widehat{\alpha}_t = \frac{N_t}{\int_0^t g(N_s)\, ds}.$$

....(5p)

5. Consider a market model defined by

$$\begin{aligned} dS_t &= \alpha S_t dt + \beta S_{t-} dN_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Here, α, β and r are known constants, and we assume that $\beta > -1$, and that $(r - \alpha)/\beta > 0$. The process N is Poisson with P-intensity λ. We know that absence of arbitrage implies that there exists a measure $Q \sim P$, known as the risk neutral martingale measure, with the property that the process

$$\frac{S_t}{B_t}$$

is a Q-martingale. From general arbitrage theory it does in fact follows that the roles of S and B can be reversed, and we cite the following standard result.

Proposition 0.1 *Assuming absence of arbitrage, there exists a measure $Q^S \sim P$ such that the process Z, defined by*

$$Z_t = \frac{B_t}{S_t}$$

is a Q^S-martingale.

You now have the following problems to solve.

(a) Compute the P-dynamics of Z. .... (2p)

(b) Determine, as explicitly as possible, the dynamics of the likelihood process L, defined by

$$L_t = \frac{dQ^S}{dP}, \quad \text{on } \mathcal{F}_t.$$

and determine the intensity λ^S of N under Q^S. .... (3p)

Good luck!