Stochastic Optimal Control with Finance Applications

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• Investment theory.

• The martingale approach.

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1. Dynamic Programming

- The basic idea.
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- The linear quadratic regulator.
Problem Formulation

\[
\max_u \mathbb{E} \left[ \int_0^T F(t, X_t, u_t) dt + \Phi(X_T) \right]
\]

subject to

\[
dX_t = \mu(t, X_t, u_t) \, dt + \sigma(t, X_t, u_t) \, dW_t
\]

\[
X_0 = x_0,
\]

\[
u_t \in U(t, X_t), \quad \forall t.
\]

We will only consider feedback control laws, i.e. controls of the form

\[
u_t = u(t, X_t)
\]

Terminology:

\[
X = \text{state variable}
\]

\[
u = \text{control variable}
\]

\[
U = \text{control constraint}
\]

Note: No state space constraints.
Main idea

• Embed the problem above in a family of problems indexed by starting point in time and space.

• Tie all these problems together by a PDE—the Hamilton Jacobi Bellman equation.

• The control problem is reduced to the problem of solving the deterministic HJB equation.
Some notation

• For any fixed vector $u \in R^k$, the functions $\mu^u$, $\sigma^u$ and $C^u$ are defined by

\[
\begin{align*}
\mu^u(t, x) &= \mu(t, x, u), \\
\sigma^u(t, x) &= \sigma(t, x, u), \\
C^u(t, x) &= \sigma(t, x, u)\sigma(t, x, u)'.
\end{align*}
\]

• For any control law $u$, the functions $\mu^u$, $\sigma^u$, $C^u(t, x)$ and $F^u(t, x)$ are defined by

\[
\begin{align*}
\mu^u(t, x) &= \mu(t, x, u(t, x)), \\
\sigma^u(t, x) &= \sigma(t, x, u(t, x)), \\
C^u(t, x) &= \sigma(t, x, u(t, x))\sigma(t, x, u(t, x))', \\
F^u(t, x) &= F(t, x, u(t, x)).
\end{align*}
\]
More notation

• For any fixed vector \( u \in \mathbb{R}^k \), the partial differential operator \( \mathcal{A}^u \) is defined by

\[
\mathcal{A}^u = \sum_{i=1}^{n} \mu^u_i(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} C^{u}_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.
\]

• For any control law \( u \), the partial differential operator \( \mathcal{A}^u \) is defined by

\[
\mathcal{A}^u = \sum_{i=1}^{n} \mu^u_i(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} C^{u}_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.
\]

• For any control law \( u \), the process \( X^u \) is the solution of the SDE

\[
dX^u_t = \mu(t, X^u_t, u_t) \, dt + \sigma(t, X^u_t, u_t) \, dW_t,
\]

where

\[
u_t = u(t, X^u_t)
\]
Embedding the problem

For every fixed \((t, x)\) the control problem \(\mathcal{P}(t, x)\) is defined as the problem to maximize

\[
E_{t,x} \left[ \int_t^T F(s, X^u_s, u_s) ds + \Phi(X^u_T) \right],
\]

given the dynamics

\[
dX^u_s = \mu(s, X^u_s, u_s) ds + \sigma(s, X^u_s, u_s) dW_s,
\]

\[
X_t = x,
\]

and the constraints

\[
u(s, y) \in U, \quad \forall (s, y) \in [t, T] \times \mathbb{R}^n.
\]

The original problem was \(\mathcal{P}(0, x_0)\).
The optimal value function

- The value function

\[ J : \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R} \]

is defined by

\[ J(t, x, u) = \mathbb{E} \left[ \int_t^T F(s, X_s^u, u_s) ds + \Phi(X_T^u) \right] \]

given the dynamics above.

- The optimal value function

\[ V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R} \]

is defined by

\[ V(t, x) = \sup_{u \in \mathcal{U}} J(t, x, u). \]

- We want to derive a PDE for \( V \).
Assumptions

We assume:

- There exists an optimal control law \( \hat{u} \).

- The optimal value function \( V \) is regular in the sense that \( V \in C^{1,2} \).

- A number of limiting procedures in the following arguments can be justified.
The Bellman Optimality Principle

Dynamic programming relies heavily on the following basic result.

**Proposition:** If \( \hat{u} \) is optimal on the time interval \([t, T]\) then it is also optimal on every subinterval \([s, T]\) with \(t \leq s \leq T\).

**Proof:** Iterated expectations.
Basic strategy

To derive the PDE do as follows:

- Fix \((t, x) \in (0, T) \times \mathbb{R}^n\).

- Choose a real number \(h\) (interpreted as a “small” time increment).

- Choose an arbitrary control law \(u\).

Now define the control law \(u^*\) by

\[
  u^*(s, y) = \begin{cases} 
    u(s, y), & (s, y) \in [t, t + h] \times \mathbb{R}^n \\
    \hat{u}(s, y), & (s, y) \in (t + h, T] \times \mathbb{R}^n.
  \end{cases}
\]

In other words, if we use \(u^*\) then we use the arbitrary control \(u\) during the time interval \([t, t + h]\), and then we switch to the optimal control law during the rest of the time period.
Basic idea

The whole idea of DynP boils down to the following procedure.

• Given the point \((t, x)\) above, we consider the following two strategies over the time interval \([t, T]\):
  
  **I:** Use the optimal law \(\hat{u}\).
  
  **II:** Use the control law \(u^*\) defined above.

• Compute the expected utilities obtained by the respective strategies.

• Using the obvious fact that Strategy I is least as good as Strategy II, and letting \(h\) tend to zero, we obtain our fundamental PDE.
Strategy values

Expected utility for strategy I:

\[ J(t, x, \hat{u}) = V(t, x) \]

Expected utility for strategy II:

- The expected utility for \([t, t + h]\) is given by

\[ E_{t,x} \left[ \int_{t}^{t+h} F(s, X_s^u, u_s) \, ds \right]. \]

- Conditional expected utility over \([t + h, T]\), given \((t, x)\):

\[ E_{t,x} \left[ V(t + h, X_{t+h}^u) \right]. \]

- Total expected utility for Strategy II is

\[ E_{t,x} \left[ \int_{t}^{t+h} F(s, X_s^u, u_s) \, ds + V(t + h, X_{t+h}^u) \right]. \]
Comparing strategies

We have trivially

$$V(t, x) \geq E_{t,x} \left[ \int_t^{t+h} F(s, X_{s}^{u}, u_s) \, ds + V(t + h, X_{t+h}^{u}) \right].$$

**Remark**

We have equality above if and only if the control law $u$ is an optimal law $\hat{u}$.

Now use Itô to obtain

$$V(t + h, X_{t+h}^{u}) = V(t, x)$$

$$+ \int_t^{t+h} \left\{ \frac{\partial V}{\partial t}(s, X_{s}^{u}) + A^u V(s, X_{s}^{u}) \right\} \, ds$$

$$+ \int_t^{t+h} \nabla_x V(s, X_{s}^{u}) \sigma^u dW_s,$$

and plug into the formula above.
We obtain

\[ E_{t,x} \left[ \int_{t}^{t+h} \left[ F(s, X_s^u, u_s) + \frac{\partial V}{\partial t}(s, X_s^u) + A^u V(s, X_s^u) \right] ds \right] \leq 0. \]

**Going to the limit:**
Divide by \( h \), move \( h \) within the expectation and let \( h \) tend to zero.
We get

\[ F(t, x, u) + \frac{\partial V}{\partial t}(t, x) + A^u V(t, x) \leq 0, \]
Recall

\[ F(t, x, u) + \frac{\partial V}{\partial t}(t, x) + A^u V(t, x) \leq 0, \]

This holds for all \( u = u(t, x) \), with equality if and only if \( u = \hat{u} \).

We thus obtain the **HJB equation**

\[
\frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{ F(t, x, u) + A^u V(t, x) \} = 0.
\]
The HJB equation

Theorem:
Under suitable regularity assumptions the following hold:

I: $V$ satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + A^u V(t, x)\} = 0,$$

$$V(T, x) = \Phi(x),$$

II: For each $(t, x) \in [0, T] \times \mathbb{R}^n$ the supremum in the HJB equation above is attained by $u = \hat{u}(t, x)$. 
Logic and problem

**Note:** We have shown that if $V$ is the optimal value function, and if $V$ is regular enough, then $V$ satisfies the HJB equation. The HJB eqn is thus derived as a necessary condition, and requires strong ad hoc regularity assumptions.

**Problem:** Suppose we have solved the HJB equation. Have we then found the optimal value function and the optimal control law?

**Answer:** Yes! This follows from the **Verification theorem**.
The Verification Theorem

Suppose that we have two functions $H(t, x)$ and $g(t, x)$, such that

- $H$ is sufficiently integrable, and solves the HJB equation
  \[
  \begin{aligned}
  \frac{\partial H}{\partial t}(t, x) + \sup_{u \in U} \{ F(t, x, u) + A^u H(t, x) \} &= 0, \\
  H(T, x) &= \Phi(x),
  \end{aligned}
  \]

- For each fixed $(t, x)$, the supremum in the expression
  \[
  \sup_{u \in U} \{ F(t, x, u) + A^u H(t, x) \}
  \]
  is attained by the choice $u = g(t, x)$.

Then the following hold.

1. The optimal value function $V$ to the control problem is given by
   \[
   V(t, x) = H(t, x).
   \]
2. There exists an optimal control law $\hat{u}$, and in fact
   \[
   \hat{u}(t, x) = g(t, x)
   \]
Handling the HJB equation

1. Consider the HJB equation for $V$.

2. Fix $(t, x) \in [0, T] \times \mathbb{R}^n$ and solve, the static optimization problem

$$\max_{u \in U} \left[ F(t, x, u) + A^u V(t, x) \right].$$

Here $u$ is the only variable, whereas $t$ and $x$ are fixed parameters. The functions $F$, $\mu$, $\sigma$ and $V$ are considered as given.

3. The optimal $\hat{u}$, will depend on $t$ and $x$, and on the function $V$ and its partial derivatives. We thus write $\hat{u}$ as

$$\hat{u} = \hat{u}(t, x; V). \quad (1)$$

4. The function $\hat{u}(t, x; V)$ is our candidate for the optimal control law, but since we do not know $V$ this description is incomplete. Therefore we substitute the expression for $\hat{u}$ into the PDE, giving us the PDE

$$\frac{\partial V}{\partial t}(t, x) + F\hat{u}(t, x) + A\hat{u}(t, x) V(t, x) = 0,$$

$$V(T, x) = \Phi(x).$$

5. Now we solve the PDE above! Then we put the solution $V$ into expression (1). Using the verification theorem we can identify $V$ as the optimal value function, and $\hat{u}$ as the optimal control law.
Making an Ansatz

• The hard work of dynamic programming consists in solving the highly nonlinear HJB equation

• There are no general analytic methods available for this, so the number of known optimal control problems with an analytic solution is very small indeed.

• In an actual case one usually tries to guess a solution, i.e. we typically make a parameterized Ansatz for $V$ then use the PDE in order to identify the parameters.

• Hint: $V$ often inherits some structural properties from the boundary function $\Phi$ as well as from the instantaneous utility function $F$.

• Most of the known solved control problems have, to some extent, been “rigged” in order to be analytically solvable.
The Linear Quadratic Regulator

\[
\min_{u \in \mathbb{R}^k} \quad E \left[ \int_0^T \{X_t'QX_t + u_t'Ru_t\} \, dt + X_T'HX_T \right],
\]

with dynamics

\[
dX_t = \{AX_t + Bu_t\} \, dt + CdW_t.
\]

We want to control a vehicle in such a way that it stays close to the origin (the terms \(x'Qx\) and \(x'Hx\)) while at the same time keeping the “energy” \(u'Ru\) small.

Here \(X_t \in \mathbb{R}^n\) and \(u_t \in \mathbb{R}^k\), and we impose no control constraints on \(u\).

The matrices \(Q, R, H, A, B\) and \(C\) are assumed to be known. We may WLOG assume that \(Q, R\) and \(H\) are symmetric, and we assume that \(R\) is positive definite (and thus invertible).
Handling the Problem

The HJB equation becomes

\[
\begin{cases}
\frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^k} \{x'Qx + u'Ru + [\nabla_x V](t, x) [Ax + Bu]\} \\
+ \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x) [CC']_{i,j} = 0, \\
V(T, x) = x'Hx.
\end{cases}
\]

For each fixed choice of \((t, x)\) we now have to solve the static unconstrained optimization problem to minimize

\[u'Ru + [\nabla_x V](t, x) [Ax + Bu].\]
The problem was:

$$\min_u \quad u' Ru + [\nabla_x V](t, x) [Ax + Bu].$$

Since $R > 0$ we set the gradient to zero and obtain

$$2u'R = -(\nabla_x V)B,$$

which gives us the optimal $u$ as

$$\hat{u} = -\frac{1}{2} R^{-1} B' (\nabla_x V)' .$$

**Note:** This is our candidate of optimal control law, but it depends on the unknown function $V$.

We now make an educated guess about the shape of $V$. 
From the boundary function $x'Hx$ and the term $x'Qx$ in the cost function we make the Ansatz

$$V(t, x) = x'P(t)x + q(t),$$

where $P(t)$ is a symmetric matrix function, and $q(t)$ is a scalar function.

With this trial solution we have,

$$\frac{\partial V}{\partial t}(t, x) = x'\dot{P}x + \dot{q},$$

$$\nabla_x V(t, x) = 2x'P,$$

$$\nabla_{xx} V(t, x) = 2P$$

$$\hat{u} = -R^{-1}B'Px.$$

Inserting these expressions into the HJB equation we get

$$x' \left\{ \dot{P} + Q - PBR^{-1}B'P + A'P + PA \right\} x$$

$$+ \dot{q} + tr[C'PC] = 0.$$
We thus get the following matrix ODE for $P$

\[
\begin{align*}
\dot{P} &= PBR^{-1}B'P - A'P - PA - Q, \\
P(T) &= H.
\end{align*}
\]

and we can integrate directly for $q$:

\[
\begin{align*}
\dot{q} &= -tr[C'PC], \\
q(T) &= 0.
\end{align*}
\]

The matrix equation is a **Riccati equation**. The equation for $q$ can then be integrated directly.

**Final Result for LQ:**

\[
\begin{align*}
V(t, x) &= x'P(t)x + \int_t^T tr[C'P(s)C]ds, \\
\hat{u}(t, x) &= -R^{-1}B'P(t)x.
\end{align*}
\]
2. Portfolio Theory

- Problem formulation.
- An extension of HJB.
- The simplest consumption-investment problem.
- The Merton fund separation results.
Recap of Basic Facts

We consider a market with $n$ assets.

- $S_t^i = \text{price of asset No } i$,
- $h_t^i = \text{units of asset No } i \text{ in portfolio}$
- $w_t^i = \text{portfolio weight on asset No } i$
- $X_t = \text{portfolio value}$
- $c_t = \text{consumption rate}$

We have the relations

$$X_t = \sum_{i=1}^{n} h_t^i S_t^i, \quad w_t^i = \frac{h_t^i S_t^i}{X_t}, \quad \sum_{i=1}^{n} w_t^i = 1.$$

**Basic equation:**
Dynamics of self financing portfolio in terms of relative weights

$$dX_t = X_t \sum_{i=1}^{n} w_t^i \frac{dS_t^i}{S_t^i} - c_t \, dt$$
**Simplest model**

Assume a scalar risky asset and a constant short rate.

\[ dS_t = \alpha S_t dt + \sigma S_t dW_t \]

\[ dB_t = r B_t dt \]

We want to maximize expected utility over time

\[
\max_{w^0, w^1, c} E \left[ \int_0^T F(t, c_t) dt + \Phi(X_T) \right]
\]

**Dynamics**

\[ dX_t = X_t \left[ u^0_t r + w^1_t \alpha \right] dt - c_t dt + w^1_t \sigma X_t dW_t, \]

**Constraints**

\[ c_t \geq 0, \ \forall t \geq 0, \]

\[ w^0_t + w^1_t = 1, \ \forall t \geq 0. \]

**Nonsense!**
What are the problems?

• We can obtain unlimited utility by simply consuming arbitrary large amounts.

• The wealth will go negative, but there is nothing in the problem formulations which prohibits this.

• We would like to impose a constraint of type $X_t \geq 0$ but this is a state constraint and DynP does not allow this.

Good News:
DynP can be generalized to handle (some) problems of this kind.
Generalized problem

Let $D$ be a nice open subset of $[0, T] \times R^n$ and consider the following problem.

$$
\max_{u \in U} \mathbb{E} \left[ \int_0^\tau F(s, X_s^u, u_s) ds + \Phi(\tau, X_\tau^u) \right].
$$

Dynamics:

$$
dX_t = \mu(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dW_t, \\
X_0 = x_0,
$$

The stopping time $\tau$ is defined by

$$
\tau = \inf \{ t \geq 0 \mid (t, X_t) \in \partial D \} \land T.
$$
Generalized HJB

**Theorem:** Given enough regularity the following hold.

1. The optimal value function satisfies

\[
\frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + A^u V(t, x)\} = 0, \quad \forall (t, x) \in D
\]

\[
V(t, x) = \Phi(t, x), \quad \forall (t, x) \in \partial D.
\]

2. We have an obvious verification theorem.
Reformulated problem

\[
\max_{c \geq 0, \ w \in \mathbb{R}} \ E \left[ \int_0^\tau F(t, c_t) \, dt + \Phi(X_T) \right]
\]

where
\[
\tau = \inf \{ t \geq 0 \mid X_t = 0 \} \land T.
\]

with notation:
\[
\begin{align*}
    w^1 &= w, \\
    w^0 &= 1 - w
\end{align*}
\]

Thus no constraint on \( w \).

Dynamics
\[
\, dX_t = w_t \left[ \alpha - r \right] X_t \, dt + \left( r X_t - c_t \right) \, dt + w \sigma X_t \, dW_t,
\]
HJB Equation

\[
\frac{\partial V}{\partial t} + \sup_{c \geq 0, w \in \mathbb{R}} \left\{ F(t, c) + wx(\alpha - r) \frac{\partial V}{\partial x} + (rx - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2} \right\} = 0,
\]

\[
V(T, x) = 0,
\]

\[
V(t, 0) = 0.
\]

We now specialize (why?) to

\[
F(t, c) = e^{-\delta t} c^\gamma,
\]

so we have to maximize

\[
e^{-\delta t} c^\gamma + wx(\alpha - r) \frac{\partial V}{\partial x} + (rx - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},
\]
Analysis of the HJB Equation

In the embedded static problem we maximize, over \( c \) and \( w \),

\[
e^{-\delta t} c^\gamma + w x (\alpha - r) \frac{\partial V}{\partial x} + (r x - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},
\]

First order conditions:

\[
\gamma c^{\gamma-1} = e^{\delta t} V_x,
\]

\[
w = \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\alpha - r}{\sigma^2},
\]

Ansatz:

\[
V(t, x) = e^{-\delta t} h(t) x^\gamma,
\]

Because of the boundary conditions, we must demand that

\[
h(T) = 0. \quad (2)
\]
Given a $V$ of this form we have (using $\cdot$ to denote the time derivative)

\[
\frac{\partial V}{\partial t} = e^{-\delta t}hx^\gamma - \delta e^{-\delta t}hx^\gamma,
\]
\[
\frac{\partial V}{\partial x} = \gamma e^{-\delta t}hx^{\gamma-1},
\]
\[
\frac{\partial^2 V}{\partial x^2} = \gamma(\gamma - 1)e^{-\delta t}hx^{\gamma-2}.
\]

giving us

\[
\hat{w}(t, x) = \frac{\alpha - r}{\sigma^2(1 - \gamma)},
\]
\[
\hat{c}(t, x) = xh(t)^{-1/(1-\gamma)}.
\]

Plug all this into HJB!
After rearrangements we obtain

\[ x^\gamma \left\{ \dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} \right\} = 0, \]

where the constants \( A \) and \( B \) are given by

\[ A = \frac{\gamma(\alpha - r)^2}{\sigma^2(1 - \gamma)} + r\gamma - \frac{1\gamma(\alpha - r)^2}{2\sigma^2(1 - \gamma)} - \delta \]
\[ B = 1 - \gamma. \]

If this equation is to hold for all \( x \) and all \( t \), then we see that \( h \) must solve the ODE

\[ \dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} = 0, \]
\[ h(T) = 0. \]

An equation of this kind is known as a \textbf{Bernoulli equation}, and it can be solved explicitly.

We are done.
Merton’s Mutual Fund Theorems

1. The case with no risk free asset

We consider \( n \) risky assets with dynamics

\[
\begin{align*}
    dS_i &= S_i \alpha_i dt + S_i \sigma_i dW, \quad i = 1, \ldots, n
\end{align*}
\]

where \( W \) is Wiener in \( \mathbb{R}^k \). On vector form:

\[
    dS = D(S) \alpha dt + D(S) \sigma dW.
\]

where

\[
    \alpha = \begin{bmatrix}
        \alpha_1 \\
        \vdots \\
        \alpha_n
    \end{bmatrix}, \quad \sigma = \begin{bmatrix}
        \sigma_1 \\
        \vdots \\
        \sigma_n
    \end{bmatrix}
\]

\( D(S) \) is the diagonal matrix

\[
    D(S) = \text{diag}[S_1, \ldots, S_n].
\]

Wealth dynamics

\[
    dX = X \omega' \alpha dt - cd t + X \omega' \sigma dW.
\]
Formal problem

\[
\max_{c, w} E \left[ \int_0^T F(t, c_t) dt \right]
\]
given the dynamics

\[
dX = X w' \alpha dt - c dt + X w' \sigma dW.
\]

and constraints

\[
e'w = 1, \quad c \geq 0.
\]

Assumptions:

- The vector \( \alpha \) and the matrix \( \sigma \) are constant and deterministic.

- The volatility matrix \( \sigma \) has full rank so \( \sigma \sigma' \) is positive definite and invertible.

Note: \( S \) does not turn up in the \( X \)-dynamics so \( V \) is of the form

\[
V(t, x, s) = V(t, x)
\]
The HJB equation is

\[
\begin{align*}
  \frac{\partial V}{\partial t}(t, x) + \sup_{e', w=1, c \geq 0} \{ F(t, c) + A^{c,w}V(t, x) \} &= 0, \\
  V(T, x) &= 0, \\
  V(t, 0) &= 0.
\end{align*}
\]

where

\[
A^{c,w}V = x w' \alpha \frac{\partial V}{\partial x} - c \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w' \Sigma w \frac{\partial^2 V}{\partial x^2},
\]

and where the matrix \( \Sigma \) is given by

\[
\Sigma = \sigma \sigma'.
\]
The HJB equation is

\[
\begin{cases}
V_t(t, x) + \sup_{w' e = 1, \ c \geq 0} \left\{ F(t, c) + (xw' \alpha - c)V_x(t, x) + \frac{1}{2}x^2w' \Sigma wV_{xx}(t, x) \right\} = 0, \\
V(T, x) = 0, \\
V(t, 0) = 0.
\end{cases}
\]

where \( \Sigma = \sigma \sigma' \).

If we relax the constraint \( w' e = 1 \), the Lagrange function for the static optimization problem is given by

\[
L = F(t, c) + (xw' \alpha - c)V_x(t, x) + \frac{1}{2}x^2w' \Sigma wV_{xx}(t, x) + \lambda (1 - w' e).
\]
\[ L = F(t, c) + (xw'\alpha - c)V_x(t, x) \]
\[ + \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x) + \lambda (1 - w'e) . \]

The first order condition for \( c \) is
\[ \frac{\partial F}{\partial c}(t, c) = V_x(t, x). \]

The first order condition for \( w \) is
\[ x\alpha'V_x + x^2V_{xx}w'\Sigma = \lambda e', \]
so we can solve for \( w \) in order to obtain
\[ \hat{w} = \Sigma^{-1} \left[ \frac{\lambda}{x^2V_{xx}} e - \frac{xV_x}{x^2V_{xx}} \alpha \right]. \]

Using the relation \( e'w = 1 \) this gives \( \lambda \) as
\[ \lambda = \frac{x^2V_{xx} + xV_xe'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}, \]
Inserting \( \lambda \) gives us, after some manipulation,\[ \hat{w} = \frac{1}{e'\Sigma^{-1}e} \Sigma^{-1}e + \frac{V_x}{x V_{xx}} \Sigma^{-1} \left[ \frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e} e - \alpha \right]. \]

We can write this as \[ \hat{w}(t) = g + Y(t)h, \]
where the fixed vectors \( g \) and \( h \) are given by
\[
\begin{align*}
g &= \frac{1}{e'\Sigma^{-1}e} \Sigma^{-1}e, \\
h &= \Sigma^{-1} \left[ \frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e} e - \alpha \right],
\end{align*}
\]
whereas \( Y \) is given by
\[ Y(t) = \frac{V_x(t, X(t))}{X(t) V_{xx}(t, X(t))}. \]
We had

$$\hat{w}(t) = g + Y(t)h,$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional “optimal portfolio line”

$$g + sh,$$

in the \((n - 1)\)-dimensional “portfolio hyperplane” \(\Delta\), where

$$\Delta = \{ w \in \mathbb{R}^n \mid e'w = 1 \}.$$

If we fix two points on the optimal portfolio line, say \(w^a = g + ah\) and \(w^b = g + bh\), then any point \(w\) on the line can be written as an affine combination of the basis points \(w^a\) and \(w^b\). An easy calculation shows that if \(w^s = g + sh\) then we can write

$$w^s = \mu w^a + (1 - \mu)w^b,$$

where

$$\mu = \frac{s - b}{a - b}.$$
Mutual Fund Theorem

There exists a family of mutual funds, given by $w^s = g + sh$, such that

1. For each fixed $s$ the portfolio $w^s$ stays fixed over time.

2. For fixed $a, b$ with $a \neq b$ the optimal portfolio $\hat{w}(t)$ is, obtained by allocating all resources between the fixed funds $w^a$ and $w^b$, i.e.

   $$\hat{w}(t) = \mu^a(t)w^a + \mu^b(t)w^b,$$

3. The relative proportions $(\mu^a, \mu^b)$ of wealth allocated to $w^a$ and $w^b$ are given by

   $$\mu^a(t) = \frac{Y(t) - b}{a - b},$$

   $$\mu^b(t) = \frac{a - Y(t)}{a - b}.$$
The case with a risk free asset

Again we consider the standard model

\[ dS = D(S)\alpha dt + D(S)\sigma dW(t), \]

We also assume the risk free asset \( B \) with dynamics

\[ dB = rB dt. \]

We denote \( B = S_0 \) and consider portfolio weights \((w_0, w_1, \ldots, w_n)'\) where \( \sum_{0}^{n} w_i = 1 \). We then eliminate \( w_0 \) by the relation

\[ w_0 = 1 - \sum_{1}^{n} w_i, \]

and use the letter \( w \) to denote the portfolio weight vector for the risky assets only. Thus we use the notation

\[ w = (w_1, \ldots, w_n)', \]

**Note:** \( w \in \mathbb{R}^n \) without constraints.
We obtain
\[ dX = X \cdot w'(\alpha - re)dt + (rX - c)dt + X \cdot w'\sigma dW, \]
where \( e = (1, 1, \ldots, 1)' \).

The HJB equation now becomes
\[
\begin{align*}
V_t(t, x) + \sup_{c \geq 0, w \in \mathbb{R}^n} \{ F(t, c) + \mathcal{A}_{c,w} V(t, x) \} &= 0, \\
V(T, x) &= 0, \\
V(t, 0) &= 0,
\end{align*}
\]
where
\[
\mathcal{A}_{c} V = x w'(\alpha - re) V_x(t, x) + (rx - c) V_x(t, x) \\
+ \frac{1}{2} x^2 w' \Sigma w V_{xx}(t, x).
\]
First order conditions

We maximize

\[ F(t, c) + xw'(\alpha - re)V_x + (rx - c)V_x + \frac{1}{2}x^2w'\Sigma wV_{xx} \]

with \( c \geq 0 \) and \( w \in \mathbb{R}^n \).

The first order conditions are

\[ \frac{\partial F}{\partial c}(t, c) = V_x(t, x), \]

\[ \hat{w} = -\frac{V_x}{xV_{xx}}\Sigma^{-1}(\alpha - re), \]

with geometrically obvious economic interpretation.
Mutual Fund Separation Theorem

1. The optimal portfolio consists of an allocation between two fixed mutual funds $w^0$ and $w^f$.

2. The fund $w^0$ consists only of the risk free asset.

3. The fund $w^f$ consists only of the risky assets, and is given by
   \[ w^f = \Sigma^{-1}(\alpha - re). \]

4. At each $t$ the optimal relative allocation of wealth between the funds is given by
   \[ \mu^f(t) = -\frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}, \]
   \[ \mu^0(t) = 1 - \mu^f(t). \]
3. The Martingale Approach

- Decoupling the wealth profile from the portfolio choice.

- Lagrange relaxation.

- Solving the general wealth problem.

- Example: Log utility.

- Example: The numeraire portfolio.

- Computing the optimal portfolio.

- The Merton fund separation theorems from a martingale perspective.
Problem Formulation

Standard model with internal filtration

\[ dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t, \]
\[ dB_t = rB_t dt. \]

Assumptions:

- Drift and diffusion terms are allowed to be arbitrary adapted processes.
- The market is complete.
- We have a given initial wealth \( x_0 \)

Problem:

\[ \max_{h \in \mathcal{H}} E^P [\Phi(X_T)] \]

where

\[ \mathcal{H} = \{\text{self financing portfolios}\} \]

given the initial wealth \( X_0 = x_0 \).
Some observations

- In a complete market, there is a unique martingale measure $Q$.

- Every claim $Z$ satisfying the budget constraint

$$e^{-rT}E^Q[Z] = x_0,$$

is attainable by an $h \in \mathcal{H}$ and vice versa.

- We can thus write our problem as

$$\max_Z E^P[\Phi(Z)]$$

subject to the constraint

$$e^{-rT}E^Q[Z] = x_0.$$

- We can forget the wealth dynamics!
Basic Ideas

Our problem was

$$\max_Z E^P [\Phi(Z)]$$

subject to

$$e^{-rT} E^Q [Z] = x_0.$$ 

Idea I:

We can **decouple** the optimal portfolio problem:

- Finding the optimal wealth profile $\hat{Z}$.
- Given $\hat{Z}$, find the replicating portfolio.

Idea II:

- Rewrite the constraint under the measure $P$.
- Use Lagrangian techniques to relax the constraint.
Lagrange formulation

Problem:
\[
\max_Z \quad E^P [\Phi(Z)]
\]
subject to
\[
e^{-rT} E^P [L_T Z] = x_0.
\]

Here \( L \) is the likelihood process, i.e.
\[
L_T = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_T
\]

The Lagrangian of this is
\[
\mathcal{L} = E^P [\Phi(Z)] + \lambda \left\{ x_0 - e^{-rT} E^P [L_T Z] \right\}
\]
i.e.
\[
\mathcal{L} = E^P [\Phi(Z) - \lambda e^{-rT} L_T Z] + \lambda x_0
\]
The optimal wealth profile

Given enough convexity and regularity we now expect, given the dual variable $\lambda$, to find the optimal $Z$ by maximizing

$$
\mathcal{L} = E^P \left[ \Phi(Z) - \lambda e^{-rT} L_T Z \right] + \lambda x_0
$$

over unconstrained $Z$, i.e. to maximize

$$
\int_\Omega \left\{ \Phi(Z(\omega)) - \lambda e^{-rT} L_T(\omega) Z(\omega) \right\} dP(\omega)
$$

This is a trivial problem!

We can simply maximize $Z(\omega)$ for each $\omega$ separately.

$$
\max_z \left\{ \Phi(z) - \lambda e^{-rT} L_T z \right\}
$$
The optimal wealth profile

Our problem:

\[
\max_z \{ \Phi(z) - \lambda e^{-r^TL_Tz} \}
\]

First order condition

\[
\Phi'(z) = \lambda e^{-r^TL_T}
\]

The optimal \( Z \) is thus given by

\[
\hat{Z} = G \left( \lambda e^{-r^TL_T} \right)
\]

where

\[
G(y) = [\Phi']^{-1} (y).
\]

The dual variable \( \lambda \) is determined by the constraint

\[
e^{-r^T}EP \left[ L_T\hat{Z} \right] = x_0.
\]
Example – log utility

Assume that
\[ \Phi(x) = \ln(x) \]

Then
\[ g(y) = \frac{1}{y} \]

Thus
\[ \hat{Z} = G \left( \lambda e^{-rT} L_T \right) = \frac{1}{\lambda} e^{rT} L_T^{-1} \]

Finally \( \lambda \) is determined by

\[ e^{-rT} E^P \left[ L_T \hat{Z} \right] = x_0. \]

i.e.
\[ e^{-rT} E^P \left[ L_T \frac{1}{\lambda} e^{rT} L_T^{-1} \right] = x_0. \]

so \( \lambda = x_0^{-1} \) and

\[ \hat{Z} = x_0 e^{rT} L_T^{-1} \]
The Numeraire Portfolio

**Standard approach:**
- Choose a fixed numeraire (portfolio) $N$.
- Find the corresponding martingale measure, i.e. find $Q^N$ s.t.
  \[
  \frac{B}{N}, \quad \text{and} \quad \frac{S}{N}
  \]
  are $Q^N$-martingales.

**Alternative approach:**
- Choose a fixed measure $Q$.
- Find numeraire $N$ such that $Q = Q^N$.

**Special case:**
- Set $Q = P$
- Find numeraire $N$ such that $Q^N = P$ i.e. such that
  \[
  \frac{B}{N}, \quad \text{and} \quad \frac{S}{N}
  \]
  are $Q^N$-martingales under the **objective** measure $P$.
- This $N$ is the **numeraire portfolio**.
Log utility and the numeraire portfolio

Definition:
The growth optimal portfolio (GOP) is the portfolio which is optimal for log utility (for arbitrary terminal date $T$).

Theorem:
Assume that $X$ is GOP. Then $X$ is the numeraire portfolio.

Proof:
We have to show that the process

$$Y_t = \frac{S_t}{X_t}$$

is a $P$ martingale. From above we know that

$$X_T = x_0 e^{rT} L_T^{-1}.$$

We also have (why?)

$$X_t = e^{-r(T-t)} E^Q \left[ X_T \mid \mathcal{F}_t \right] = e^{-r(T-t)} E^P \left[ \frac{X_T L_T}{L_t} \mid \mathcal{F}_t \right]$$

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Thus

\[ X_t = e^{-r(T-t)} E^P \left[ \frac{X_T L_T}{L_t} \bigg| \mathcal{F}_t \right] \]

\[ = e^{-r(T-t)} E^Q \left[ \frac{x_0 e^{rT} L_T}{L_T L_t} \bigg| \mathcal{F}_t \right] = x_0 e^{rt} L_t^{-1}. \]

as expected.

Thus

\[ \frac{S_t}{X_t} = x_0^{-1} e^{-rt} S_t L_t \]

which is a \( P \) martingale, since \( x_0^{-1} e^{-rt} S_t \) is a \( Q \) martingale.
Back to general case: Computing $L_T$

We recall

$$\hat{Z} = G \left( \lambda e^{-r^T L_T} \right).$$

The likelihood process $L$ is computed by using Girsanov. We recall

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,$$

We know from Girsanov that

$$dL_t = L_t\varphi^*_t dW_t$$

so

$$dW_t = \varphi_t dt + dW^Q_t$$

where $W^Q$ is $Q$-Wiener.

Thus

$$dS_t = D(S_t) \{\alpha_t + \sigma_t\varphi_t\} dt + D(S_t)\sigma_t dW^Q_t,$$
Computing $L_T$, continued

Recall

\[ dS_t = D(S_t) \{ \alpha_t + \sigma_t \varphi_t \} \, dt + D(S_t) \sigma_t dW_t^Q, \]

The kernel $\varphi$ is determined by the martingale measure condition

\[ \alpha_t + \sigma_t \varphi_t = r \]

where

\[ r = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix} \]

Market completeness implies that $\sigma_t$ is invertible so

\[ \varphi_t = \sigma_t^{-1} \{ r - \alpha_t \} \]

and

\[ L_T = \exp \left( \int_0^T \varphi_t dW_t - \frac{1}{2} \int_0^T \| \varphi_t \|^2 \, dt \right) \]
Finding the optimal portfolio

• We can easily compute the optimal wealth profile.

• How do we compute the optimal portfolio?

Recall:

\[
dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,
\]

wealth dynamics

\[
dx_t = h_t^B dB_t + h_t^S dS_t
\]
or

\[
dx_t = X_tu_t^B r dt + X_tu_t^S D(S_t)^{-1} dS_t
\]

where

\[
h^S = (h^1, \ldots, h^n), \quad u^S = (u^1, \ldots, u^n)
\]

Assume for simplicity that \( r = 0 \) or consider normalized prices.
Recall wealth dynamics

\[ dX_t = h_t^S dS_t \]

alternatively

\[ dX_t = h_t^S D(S_t) \sigma_t dW_t^Q \]

alternatively

\[ dX_t = X_t u_t^S \sigma_t dW_t^Q \]

Obvious facts:

- \( X \) is a \( Q \) martingale.

- \( X_T = \hat{Z} \)

Thus the optimal wealth process is determined by

\[ X_t = E^Q \left[ \hat{Z} \middle| \mathcal{F}_t \right] \]
Recall

\[ X_t = E^Q \left[ \hat{Z} \bigg| \mathcal{F}_t \right] \]

Martingale representation theorem gives us

\[ dX_t = \xi_t dW_t^Q, \]

but also

\[
\begin{align*}
    dX_t &= X_t u^S_t \sigma_t dW_t^Q \\
    dX_t &= h^S_t D(S_t) \sigma_t dW_t^Q
\end{align*}
\]

Thus \( u^S \) and \( h^S_t \) are determined by

\[
\begin{align*}
    u^S_t &= \frac{1}{X_t} \xi_t \sigma_t^{-1} \\
    h^S_t &= \xi_t \sigma_t^{-1} D(S_t)^{-1}
\end{align*}
\]

and

\[
\begin{align*}
    u^B_t &= 1 - u^S_t e, \\
    h^B_t &= X_t - h^S_t S_t
\end{align*}
\]
How do we find $\xi$?

Recall

\[
X_t = E^Q \left[ \hat{Z} \middle| \mathcal{F}_t \right]
\]

\[
dX_t = \xi_t dW_t^Q,
\]

We need to compute $\xi$.

In a Markovian framework this follows directly from the Itô formula.

Recall

\[
\hat{Z} = H(L_T) = G(\lambda L_T)
\]

where

\[
G = [\Phi']^{-1}
\]

and

\[
dL_t = L_t \varphi^*_t dW_t,
\]

\[
dW = \varphi dt + dW_t^Q
\]

so

\[
dL_t = L_t \|\varphi_t\|^2 dt + L_t \varphi_t^* dW_t^Q
\]
Finding $\xi$

If the model is Markovian we have

\[ \alpha_t = \alpha(S_t), \quad \sigma_t = \sigma(S_t), \quad \varphi_t = \sigma(S_t)^{-1} \{ \alpha(S_t) - r \} \]

so

\[ X_t = E^Q \left[ H(L_T) \mid F_t \right] \]
\[ dS_t = D(S_t)\sigma(S_t)dW_t^Q, \]
\[ dL_t = L_t\|\varphi(S_t)\|^2 dt + L_t\varphi(S_t)^*dW_t^Q \]

Thus we have

\[ X_t = F(t, S_t, L_t) \]

where, by the Kolmogorov backward equation

\[
F_t + L\|\varphi\|^2 F_L + \frac{1}{2}L^2\|\varphi\|^2 F_{LL} + \frac{1}{2}tr \{ \sigma F_{ss} \varphi^* \} = 0, \quad F(T, s, L) = H(L)
\]
Finding $\xi$, contd.

We had

$$X_t = F(t, S_t, L_t)$$

and Itô gives us

$$dX_t = \{F_{SD}(S_t) \sigma(S_t) + F_{L_L} L_t \varphi^*(S_t)\} dW_t$$

Thus

$$\xi_t = F_{SD}(S_t) \sigma(S_t) + F_{L_L} L_t \varphi^*(S_t).$$

and

$$u_t^S = \frac{1}{X_t} \xi_t \sigma(S_t)^{-1}$$

$$h_t^S = \xi_t \sigma(S_t)^{-1} D(S_t)^{-1}.$$
Mutual Funds – Martingale Version

We now assume constant parameters

\[ \alpha(s) = \alpha, \quad \sigma(s) = \sigma, \quad \varphi(s) = \varphi \]

We recall

\[ X_t = E^Q [ H(L_T) | \mathcal{F}_t ] \]

\[ dL_t = L_t \| \varphi \|^2 dt + L_t \varphi^* dW^Q_t \]

Now \( L \) is Markov so we have (without any \( S \))

\[ X_t = F(t, L_t) \]

Thus

\[ \xi_t = F_LL_t \varphi^*, \quad u_t^S = \frac{F_LL_t}{X_t} \varphi^* \sigma^{-1} \]

and we have fund separation with the fixed risky fund given by

\[ w = \varphi^* \sigma^{-1} = \{ r^* - \alpha^* \} \{ \sigma \sigma^* \}^{-1}. \]
3. Filtering theory

- Motivational problem.
- The Innovations process.
- The non-linear FKK filtering equations.
- The Wonham filter.
- The Kalman filter.
An investment problem with stochastic rate of return

Model:

\[ dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t \]

\( W \) is scalar and \( Y \) is some factor process. We assume that \((S, Y)\) is Markov and adapted to the filtration \( \mathbb{F} \).

Wealth dynamics

\[ dX_t = X_t [r + u_t (\alpha - r)] dt + u_t X_t \sigma dW_t \]

Objective:

\[ \max_u E^P [\Phi(X_T)] \]

Information structure:

- Complete information: We observe \( S \) and \( Y \), so \( u \in \mathbb{F} \)

- Incomplete information: We only observe \( S \), so \( u \in \mathbb{F}^S \). We need filtering theory.
Filtering Theory – Setup

Given some filtration $\mathcal{F}$:

$$
\begin{align*}
    dY_t &= a_t dt + dM_t \\
    dZ_t &= b_t dt + dW_t
\end{align*}
$$

Here all processes are $\mathcal{F}$ adapted and

$$
\begin{align*}
    Y &= \text{signal process}, \\
    Z &= \text{observation process}, \\
    M &= \text{martingale w.r.t. } \mathcal{F} \\
    W &= \text{Wiener w.r.t. } \mathcal{F}
\end{align*}
$$

We assume (for the moment) that $M$ and $W$ are independent.

Problem:

Compute (recursively) the filter estimate

$$
\hat{Y}_t = \Pi_t [Y] = E \left[ Y_t \mid \mathcal{F}_t^Z \right]
$$
The innovations process

Recall:
\[ dZ_T = b_t \, dt + dW_t \]

Our best guess of \( b_t \) is \( \hat{b}_t \), so the genuinely new information should be
\[ dZ_t - \hat{b}_t \, dt \]

The innovations process \( \nu \) is defined by
\[ \nu_t = dZ_t - \hat{b}_t \, dt \]

Theorem: The process \( \nu \) is \( F^Z \)-Wiener.

Proof: By Levy it is enough to show that
- \( \nu \) is an \( F^Z \) martingale.
- \( \nu_t^2 - t \) is an \( F^Z \) martingale.
I. \( \nu \) is an \( \mathcal{F}^Z \) martingale:

From definition we have

\[
d\nu_t = \left( b_t - \hat{b}_t \right) dt + dW_t
\]

so

\[
E_s^Z [\nu_t - \nu_s] = \int_s^t E_u^Z \left[ b_u - \hat{b}_u \right] du + E_s^Z [W_t - W_s]
\]

\[= \int_s^t E_u^Z \left[ E_u^Z \left[ b_u - \hat{b}_u \right] \right] du + E_s^Z [E_s [W_t - W_s]] = 0
\]

I. \( \nu_t^2 - t \) is an \( \mathcal{F}^Z \) martingale:

From Itô we have

\[
d\nu_t^2 = 2\nu_t d\nu_t + (d\nu_t)^2
\]

Here \( d\nu \) is a martingale increment and from (3) it follows that \( (d\nu_t)^2 = dt \).
**Remark 1:**
The innovations process gives us a Gram-Schmidt orthogonalization of the increasing family of Hilbert spaces

\[ L^2(\mathcal{F}_t^Z); \quad t \geq 0. \]

**Remark 2:**
The use of Itô above requires general semimartingale integration theory, since we do not know a priori that \( \nu \) is Wiener.
Filter dynamics

From the $Y$ dynamics we guess that

$$d\hat{Y}_t = \hat{a}_t dt + \text{martingale}$$

**Definition:** $dm_t = d\hat{Y}_t - \hat{a}_t dt$.

**Proposition:** $m$ is an $\mathcal{F}_t^Z$ martingale.

**Proof:**

$$E^Z_s [m_t - m_s] = E^Z_s [\hat{Y}_t - \hat{Y}_s] - E^Z_s \left[ \int_s^t \hat{a}_u du \right]$$

$$= E^Z_s [Y_t - Y_s] - E^Z_s \left[ \int_s^t \hat{a}_u du \right]$$

$$= E^Z_s [M_t - M_s] - E^Z_s \left[ \int_s^t (a_u - \hat{a}_u) du \right]$$

$$= E^Z_s [E_s [M_t - M_s]] - E^Z_s \left[ \int_s^t E^Z_u [a_u - \hat{a}_u] du \right] = 0$$

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Filter dynamics

We now have the filter dynamics

\[ d\hat{Y}_t = \hat{a}_t dt + dm_t \]

where \( m \) is an \( \mathcal{F}_t^Z \) martingale.

If the innovations hypothesis

\[ \mathcal{F}_t^Z = \mathcal{F}_t^\nu \]

is true, then the martingale representation theorem would give us an \( \mathcal{F}_t^Z \) adapted process \( h \) such that

\[ dm_t = h_t d\nu_t \]

The innovations hypothesis is not generally correct but FKK have proved that in fact (4) is always true.
Filter dynamics

We thus have the filter dynamics

\[ d\hat{Y}_t = \hat{a}_t dt + h_t d\nu_t \]

and it remains to determine the gain process \( h \).

**Proposition:** The process \( h \) is given by

\[ h_t = \hat{Y}_t b_t - \hat{Y}_t \hat{b}_t \]

We give a slightly heuristic proof.
Proof sketch

From Itô we have

\[ d \left( Y_t Z_t \right) = Y_t b_t dt + Y_t dW_t + Z_t a_t dt + Z_t dM_t \]

using

\[ d \hat{Y}_t = \hat{a}_t dt + h_t \, d\nu_t \]

and

\[ dZ_t = \hat{b}_t dt + d\nu_t \]

we have

\[ d \left( \hat{Y}_t Z_t \right) = \hat{Y}_t \hat{b}_t dt + \hat{Y}_t d\nu_t + Z_t \hat{a}_t dt + Z_t h_t d\nu_t + h_t dt \]

Formally we also should have

\[ E \left[ d \left( Y_t Z_t \right) - d \left( \hat{Y}_t X_t \right) \mid \mathcal{F}_t^Z \right] = 0 \]

which gives us

\[ \left( \hat{Y}_t b_t - \hat{Y}_t \hat{b}_t - h_t \right) dt = 0. \]
The filter equations

For the model

\[ dY_t = a_t dt + dM_t \]
\[ dZ_T = b_t dt + dW_t \]

where \( M \) and \( W \) are independent, we have the FKK non-linear filter equations

\[ d\hat{Y}_t = \hat{a}_t dt + \{ \hat{Y}_t b_t - \hat{Y}_t \hat{b}_t \} \, d\nu_t \]
\[ d\nu_t = dZ_t - \hat{b}_t dt \]

Remark: It is easy to see that

\[ h_t = E \left[ (Y_t - \hat{Y}_t) (b_t - \hat{b}_t) \bigg| \mathcal{F}_t^Z \right] \]
The general filter equations

For the model

\[ dY_t = a_t dt + dM_t \]
\[ dZ_T = b_t dt + \sigma_t dW_t \]

where

- The process \( \sigma \) is \( \mathcal{F}_t^Z \) adapted and positive.
- There is no assumption of independence between \( M \) and \( W \).

we have the filter

\[ d\hat{Y}_t = \hat{a}_t dt + \left[ \hat{D}_t + \frac{1}{\sigma_t} \left\{ \hat{Y}_t b_t - \hat{Y}_t \hat{b}_t \right\} \right] d\nu_t \]
\[ d\nu_t = \frac{1}{\sigma_t} \left\{ dZ_t - \hat{b}_t dt \right\} \]
\[ dD_t = \frac{d\langle M, W \rangle_t}{dt} \]
Comment on $\langle M, W \rangle$

This requires semimartingale theory but there are two simple cases

• If $M$ is Wiener then

$$d\langle M, W \rangle_t = dM_t dW_t$$

with usual multiplication rules.

• If $M$ is a pure jump process then

$$d\langle M, W \rangle_t = 0.$$
Filtering a Markov process

Assume that $Y$ is Markov with generator $G$. We want to compute $\Pi_t [f(Y_t)]$, for some nice function $f$. Dynkin’s formula gives us

$$df(Y_t) = (Gf)(Y_t)dt + dM_t$$

Assume that the observations are

$$dZ_t = b(Y_t)dt + dW_t$$

where $W$ is independent of $Y$.

The filter equations are now

$$d\Pi_t [f] = \Pi_t [Gf] dt + \{\Pi_t [fb] - \Pi_t [f] \Pi_t [b]\} d\nu_t$$

$$d\nu_t = dZ_t - \Pi_t [b] dt$$

Remark: To obtain $d\Pi_t [f]$ we need $\Pi_t [fb]$ and $\Pi_t [b]$. This leads generically to an infinite dimensional system of filter equations.
On the filter dimension

\[ d\Pi_t[f] = \Pi_t[Gf] \, dt + \{\Pi_t[fb] - \Pi_t[f] \Pi_t[b]\} \, d\nu_t \]

- To obtain \(d\Pi_t[f]\) we need \(\Pi_t[fb]\) and \(\Pi_t[b]\).
- Thus we apply the FKK equations to \(Gf\) and \(b\).
- This leads to new filter estimates to determine and generically to an infinite dimensional system of filter equations.
- The filter equations are really equations for the entire conditional distribution of \(Y\).
- You can only expect the filter to be finite when the conditional distribution of \(Y\) is finitely parameterized.
- There are only very few examples of finite dimensional filters.
- The most well known finite filters are the Wonham and the Kalman filters.
The Wonham filter

Assume that $Y$ is a continuous time Markov chain on the state space $\{1, \ldots, n\}$ with (constant) generator matrix $H$. Define the indicator processes by

$$\delta_i(t) = I\{Y_t = i\}, \quad i = 1, \ldots, n.$$ 

Dynkin’s formula gives us

$$d\delta^i_t = \sum_j H(j, i) \delta_j dt + dM^i_t, \quad i = 1, \ldots, n.$$ 

Observations are

$$dZ_t = b(Y_t)dt + dW_t.$$ 

Filter equations:

$$d\Pi_t [\delta_i] = \sum_j H(j, i) \Pi_t [\delta_j] dt + \{\Pi_t [\delta_i b] - \Pi_t [\delta_i] \Pi_t [b]\} d\nu_t$$

$$d\nu_t = dZ_t - \Pi_t [b] dt$$
We note that

\[ b(Y_t) = \sum_i b(i) \delta_i(t) \]

so

\[ \Pi_t [\delta_i b] = b(i) \Pi_t [\delta_i], \]
\[ \Pi_t [b] = \sum_j b(j) \Pi_t [\delta_j] \]

We finally have the Wonham filter

\[ d\hat{\delta}_i = \sum_j H(j, i) \hat{\delta}_j dt + \left\{ b(i) \hat{\delta}_i - \hat{\delta}_i \sum_j b(j) \hat{\delta}_j \right\} d\nu_t, \]
\[ d\nu_t = dZ_t - \sum_j b(j) \hat{\delta}_j dt \]
The Kalman filter

\[ dY_t = aY_t dt + cdV_t, \]
\[ dZ_t = Y_t dt + dW_t \]

\( W \) and \( V \) are independent Wiener

FKK gives us

\[ d\Pi_t [Y] = a\Pi_t [Y] dt + \left\{ \Pi_t [Y^2] - (\Pi_t [Y])^2 \right\} d\nu_t \]
\[ d\nu_t = dZ_t - \Pi_t [Y] dt \]

We need \( \Pi_t [Y^2] \), so use Itô to get write

\[ dY_t^2 = \left\{ 2aY_t^2 + c^2 \right\} dt + 2cY_t dV_t \]

From FKK:

\[ d\Pi_t [Y^2] = \left\{ 2a\Pi_t [Y^2] + c^2 \right\} dt \]
\[ + \left\{ \Pi_t [Y^3] - \Pi_t [Y^2] \Pi_t [Y] \right\} d\nu_t \]

Now we need \( \Pi_t [Y^3] \)! Etc!
Define the conditional error variance by

\[ H_t = \Pi_t \left[ \left( \Pi_t [Y_t] - \Pi_t [Y] \right)^2 \right] = \Pi_t [Y^2] - (\Pi_t [Y])^2 \]

Itô gives us

\[ d (\Pi_t [Y])^2 = \left[ 2a (\Pi_t [Y])^2 + H^2 \right] dt + 2\Pi_t [Y] H d\nu_t \]

and Itô again

\[
\begin{align*}
    dH_t &= \left\{ 2aH_t + c^2 - H_t^2 \right\} dt \\
    &\quad + \left\{ \Pi_t [Y^3] - 3\Pi_t [Y^2] \Pi_t [Y] + 2 (\Pi_t [Y])^3 \right\} d\nu_t
\end{align*}
\]

In this particular case we know (why?) that the distribution of \( Y \) conditional on \( Z \) is Gaussian!

Thus we have

\[ \Pi_t [Y^3] = 3\Pi_t [Y^2] \Pi_t [Y] - 2 (\Pi_t [Y])^3 \]

so \( H \) is deterministic (as expected).
The Kalman filter

Model:

\[ dY_t = aY_t dt + cdV_t, \]
\[ dZ_t = Y_t dt + dW_t \]

Filter:

\[ d\Pi_t [Y] = a\Pi_t [Y] dt + H_t d\nu_t \]
\[ \dot{H}_t = 2aH_t + c^2 - H_t^2 \]
\[ d\nu_t = dZ_t - \Pi_t [Y] dt \]

\[ H_t = \Pi_t \left[ \left( \Pi_t [Y_t] - \Pi_t [Y] \right)^2 \right] \]

**Remark:** Because of the Gaussian structure, the conditional distribution evolves on a two dimensional submanifold. Hence a two dimensional filter.
Optimal investment with stochastic rate of return

- A market model with a stochastic rate of return.
- Optimal portfolios under complete information.
- Optimal portfolios under partial information.
An investment problem with stochastic rate of return

Model:

\[ dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t \]

\( W \) is scalar and \( Y \) is some factor process. We assume that \((S, Y)\) is Markov and adapted to the filtration \( \mathcal{F} \).

Wealth dynamics

\[ dX_t = X_t \{ r + u_t [\alpha(Y_t) - r] \} dt + u_t X_t \sigma dW_t \]

Objective:

\[ \max_u E^P [X_T^\gamma] \]

We assume that \( Y \) is a Markov process. with generator \( A \). We will treat several cases.
A. Full information, $Y$ and $W$ independent.

The HJB equation for $F(t, x, y)$ becomes

$$F_t + \sup_u \left\{ u \left[ \alpha - r \right] x F_x + r x F_x + \frac{1}{2} u^2 x^2 \sigma^2 F_{xx} \right\} + \mathcal{A}F = 0$$

where $G$ operates on the $y$-variable. Obvious boundary condition

$$F(t, x, y) = x^\gamma$$

First order condition gives us:

$$\hat{u} = \frac{r - \alpha}{x \sigma^2} \cdot \frac{F_x}{F_{xx}}$$

Plug into HJB:

$$F_t - \frac{(\alpha - r)^2}{2 \sigma^2} \frac{F^2_x}{F_{xx}} + r x F_x + \mathcal{A}F = 0$$
Ansatz:

\[ F(T, x, y) = x^\gamma G(t, y) \]

\[ F_t = x^\gamma G_t, \quad F_x = \gamma x^{\gamma - 1} G, \quad F_{xx} = \gamma (\gamma - 1) x^{\gamma - 2} G \]

Plug into HJB:

\[ x^\gamma G_t + x^\gamma \frac{(\alpha - r)^2}{2\sigma^2} \beta G + r \gamma x^\gamma G + x^\gamma AG = 0 \]

where \( \beta = \gamma / (\gamma - 1) \).

\[ G_t(t, y) + H(y)G(t, y) + AG(t, y) = 0, \]
\[ G(T, y) = 1. \]

Here

\[ H(y) = r \gamma - \frac{[\alpha(y) - r]^2}{2\sigma^2} \beta \]

Kolmogorov gives us

\[ H(y) = E_t,y \left[ e^{\int_t^T H(Y_s)ds} \right] \]
B. Full information, $Y$ and $W$ dependent.

Now we allow for dependence but restrict $Y$ to dynamics of the form.

\[
\begin{align*}
    dS_t &= S_t \alpha(Y_t) dt + S_t \sigma dW_t \\
    dX_t &= X_t \{r + u_t [\alpha(Y_t) - r]\} dt + u_t X_t \sigma dW_t \\
    dY_t &= a(Y_t) dt + b(Y_t) dW_t
\end{align*}
\]

with the same Wiener process $W$ driving both $S$ and $Y$. The imperfectly correlated case is a bit more messy but can also be handled.

HJB Equation for $F(t, x, y)$:

\[
\begin{align*}
    F_t + a F_y + \frac{1}{2} b^2 F_{yy} + rx F_x \\
    + \sup_{u} \left\{ u [\alpha - r] x F_x + \frac{1}{2} u^2 x^2 \sigma^2 F_{xx} + u x b \sigma F_{xy} \right\} = 0.
\end{align*}
\]
Ansatz.

HJB:

\[ F_t + a F_y + \frac{1}{2} b^2 F_{yy} + r x F_x \\
+ \sup_u \left\{ u \left[ \alpha - r \right] x F_x + \frac{1}{2} u^2 x^2 \sigma^2 F_{xx} + u x b \sigma F_{xy} \right\} = 0. \]

After a lot of thinking we make the Ansatz

\[ F(t, x, y) = x^\gamma h^{1-\gamma}(t, y) \]

and then it is not hard to see that \( h \) satisfies a standard parabolic PDE. (Plug Ansatz into HJB).

See Zariphopoulou for details and more complicated cases.
C. Partial information.

Model:

\[ dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t \]

Assumption: \( Y \) cannot be observed directly.

Requirement: The control \( u \) must be \( \mathcal{F}_t^S \) adapted. We thus have a partially observed system.

Idea: Project the \( S \) dynamics onto the smaller \( \mathcal{F}_t^S \) filtration and add filter equations in order to reduce the problem to the completely observable case.

Set \( Z = \ln S \) and note (why?) that \( \mathcal{F}_t^Z = \mathcal{F}_t^S \). We have

\[ dZ_t = \left\{ \alpha(Y_t) - \frac{1}{2} \sigma^2 \right\} dt + \sigma dW_t \]
Projecting onto the $S$-filtration

\[ dZ_t = \left\{ \alpha(Y_t) - \frac{1}{2} \sigma^2 \right\} dt + \sigma dW_t \]

From filtering theory we know that

\[ dZ_t = \left\{ \Pi_t [\alpha] - \frac{1}{2} \sigma^2 \right\} dt + \sigma d\nu_t \]

where $\nu$ is $\mathcal{F}_t^S$-Wiener and

\[ \Pi_t [\alpha] = E \left[ \alpha(Y_t) | \mathcal{F}_t^S \right] \]

We thus have the following $S$ dynamics on the $S$ filtration

\[ dS_t = S_t \Pi_t [\alpha] dt + S_t \sigma d\nu_t \]

and wealth dynamics

\[ dX_t = X_t \left\{ r + u_t (\Pi_t [\alpha] - r) \right\} dt + u_t X_t \sigma d\nu_t \]
Reformulated problem

We now have the problem

$$\max_u E \left[ X_T^\gamma \right]$$

for $Z$-adapted controls given wealth dynamics

$$dX_t = X_t \left\{ r + u_t \left( \Pi_t [\alpha] - r \right) \right\} dt + u_t X_t \sigma d\nu_t$$

If we now can model $Y$ such that the (linear!) observation dynamics for $Z$ will produce a finite filter vector $\pi$, then we are back in the completely observable case with $Y$ replaced by $\pi$. and observation equation

We need a finite dimensional filter!

Two choices for $Y$

- Linear $Y$ dynamics. This will give us the Kalman filter. See Brendle
- $Y$ as a Markov chain. This will give us the Wonham filter. See Bäuerle and Rieder.
Kalman case (Brendle)

Assume that

\[ dS_t = Y_t S_t dt + S_t \sigma dW_t \]

with \( Y \) dynamics

\[ dY_t = a Y_t dt + c dV_t \]

where \( W \) and \( V \) are independent. Observations:

\[ dZ_t = \left\{ Y_t - \frac{1}{2} \sigma^2 \right\} dt + \sigma dW_t \]

We have a standard Kalman filter.

Wealth dynamics

\[ dX_t = X_t \left\{ r + u_t \left( \hat{Y}_t - r \right) \right\} dt + u_t X_t \sigma d\nu_t \]
Kalman case, solution.

\[
\max_u E \left[ X_T^{\gamma} \right]
\]

\[
dX_t = X_t \left\{ r + u_t \left( \hat{Y}_t - r \right) \right\} dt + u_t X_t \sigma d\nu_t
\]

\[
d\hat{Y}_t = \left\{ \hat{Y}_t - \frac{1}{2} \sigma^2 \right\} dt + H_t \sigma d\nu_t
\]

where \( H \) is deterministic and given by a Riccatti equation.

We are back in standard completely observable case with state variables \( X \) and \( \hat{Y} \).

Thus the optimal value function is of the form

\[
F(t, x, \hat{y}) = x^{\gamma} h^{1-\gamma}(t, \hat{y})
\]

where \( h \) solves a parabolic PDE and can be computed explicitly.