SF3940– PROBABILITY THEORY

SPRING 2016

HOMEWORK 2

DUE MARCH 14, 2016

You must be able to explain the following concepts:

- Random variables, distribution function, Lebesgue integral, Fubini's theorem, expected value, independence, convergence in probability, convergence almost surely

Solve at least *four* of the following problems *and* one problems of your own choice (either one of the remaining two below or one problems from the book you are using).

PROBLEM 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A_1, \dots, A_n$ be sets in $\mathcal{F}$ and $A = \cup_{k=1}^n A_k$.

(a) Show $I_A = 1 - \prod_{k=1}^n (1 - I_{A_k})$.

(b) Take expected value on both sides and derive the *inclusion-exclusion formula.*

(c) Show that $I_A \leq \sum_{k=1}^n I_{A_k}$ and derive the Bonferroni inequalities

$$
\begin{aligned}
P(A) &\leq \sum_{k=1}^n P(A_k), \\
P(A) &\geq \sum_{k=1}^n P(A_k) - \sum_{k<l} P(A_k \cap A_l), \\
P(A) &\leq \sum_{k=1}^n P(A_k) - \sum_{k<l} P(A_k \cap A_l) + \sum_{k<l<m} P(A_k \cap A_l \cap A_m), \\
&\vdots \text{ etc}
\end{aligned}
$$

PROBLEM 2. Show that $X_1, X_2, \dots$ are independent if $\sigma(X_1, \dots, X_{n-1})$ and $\sigma(X_n)$ are independent for each n.

PROBLEM 3. Show that for a nonnegative and integrable random variable X

$$E[X] = \int_0^\infty P\{X > t\} dt.$$

PROBLEM 4. (a) For a counting random variable $X = \sum_{k=1}^n I_{A_k}$, give a formula for the variance of X in terms of $P\{A_k\}$ and $P\{A_k \cap A_l\}$.

(b) If k balls are put at random into n boxes, what is the variance for $X =$ the number of empty boxes?

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PROBLEM 5. A function $\varphi : \Omega \to \mathbb{R}$ is *simple* if

$$\varphi(\omega) = \sum_{k=1}^{n} c_k I\{\omega \in A_k\},$$

where c_k are real numbers and $A_k \in \mathcal{F}$.

(a) Show that the class of $\mathcal{F}$-measurable functions is the smallest class containing the simple functions and closed under point wise limits.

(b) Use (a) to show that Y is measurable with respect to $\sigma(X)$ if and only if $Y = f(X)$ for some measurable function $f : \mathbb{R} \to \mathbb{R}$.

(c) To get a constructive proof of (b), note that

$$\{\omega : m2^{-n} \le Y < (m+1)2^{-n}\} = \{\omega : X \in B_{mn}\}$$

for some Borel sets B_{mn} and set $f_n(x) = m2^{-n}$ for $x \in B_{mn}$. Then show that $f_n(x) \to f(x)$ and $Y = f(X)$.

PROBLEM 6. Let $d(X, Y)$ be the infimum of those positive ϵ for which $P\{|X - Y| \ge \epsilon\} \le \epsilon$.

(a) Show that $d(X, Y) = 0$ off $X = Y$ with probability 1. Identify random variables that are equal with probability 1 and show that d is a metric on the resulting space.

(b) Show that $X_n \to X$ in probability iff $d(X_n, X) \to 0$.

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