SF3940– PROBABILITY THEORY

SPRING 2016

HOMEWORK 3

DUE MARCH 14, 2016

You must be able to explain the following concepts:

- Borel Cantelli's Lemma I and II, Kolmogorov's 0-1 Law, Kolmogorov's Three Series Theorem, The Weak Law of Large Numbers, The Strong Law of Large Numbers, The Glivenko-Cantelli Theorem, Applications of the SLLN.

Solve at least *three* of the following problems *and* one problems of your own choice (either one of the remaining two below or two problems form the book you are using).

PROBLEM 1. Let $\{U_n\}$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ and suppose that U_n has uniform distribution on $(0,1)$ for each n. Put

$$\begin{aligned} X_1 &= U_1, \\ X_n &= (1-U_1)\cdots(1-U_{n-1})U_n, \quad n \geq 2. \end{aligned}$$

(a) Show that $\sum_n X_n = 1$ almost surely and conclude that $X_n \to 0$ almost surely.

(b) Show that $\frac{1}{n}\log X_n$ converges almost surely and determine the limit.

PROBLEM 2. Consider a probability space $(\Omega, \mathcal{F}, P)$ where, for each $n \geq 1$, the random variables X_n have a standard Gaussian distribution in n-dimensions. Find a sequence c_n of real numbers such that $\frac{1}{c_n}\|X_n\| \to 1$ almost surely.
This shows that in high dimensions the Gaussian distribution is concentrated on a sphere with radius c_n.

PROBLEM 3. Consider an investor that can invest in risk-free bonds and risky stocks. The risk-free bond gives a return of e^r each year, on each invested krona, whereas the stock gives a stochastic return R_n at the end of the nth year. It is assumed that $\{R_n\}$ are iid and non-negative. If $P_0 = 1$ and a constant fraction α is invested in the risky stock each year, then the value P_n of the portfolio at the end of each year is

$$P_n = (1-\alpha)P_{n-1}e^r + \alpha P_{n-1}R_n.$$

Suppose $E[R_n^2] < \infty$ and $E[R_n^{-2}] < \infty$.

(a) Show that $\frac{1}{n}\log P_n \to u(\alpha)$, almost surely.

(b) Show that $u(\alpha)$ is concave.

(c) Give conditions on the distribution of R_n that guarantee that the optimal α is in $(0,1)$.

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PROBLEM 4. Linear processes are constructed as follows. Let $\{Z_n : n \in \mathbb{Z}\}$ be a sequence of iid random variables and $\{c_n : n \in \mathbb{Z}\}$ be deterministic constants. Then the linear process is

$$X_n = \sum_{k\in\mathbb{Z}} c_k Z_{n-k}.$$

(a) Suppose $E[Z] = 0$ and $E[Z^2] < \infty$. Determine necessary and sufficient conditions on the sequence $\{c_n\}$, such that X_0 is finite almost surely.

(b) Suppose $P(Z < -x) = P(Z > x) = \frac{1}{2}(1 + x)^{-\alpha}$, for all $x > 0$ and $\alpha \in (0, 1)$. Determine necessary and sufficient conditions on the sequence $\{c_n\}$, such that X_0 is finite almost surely.

PROBLEM 5. (Records) Let $X_1, X_2, \ldots$ be iid random variables on a probability space $(\Omega, \mathcal{F}, P)$ with continuous distribution function. Let $B = \cup_{m,n}\{\omega : X_n(\omega) = X_m(\omega)\}$ be the set where ties occur and remove B from Ω. That is, we consider $\Omega_0 = \Omega \setminus B$ as the sample space.

Let $\Pi^n(\omega) = (\Pi^n_1(\omega), \ldots, \Pi^n_n(\omega))$ be the permutation $(\pi_1, \ldots, \pi_n)$ of $(1, \ldots, n)$ for which $X_{\Pi^n_1} < \cdots < X_{\Pi^n_n}$ and let R_n be the rank of X_n among $X_1, \ldots, X_n$. That is, $R_n = r$ if $X_k < X_n$ for exactly $r - 1$ values of $k \le n$.

(a) Show that Π^n is uniformly distributed among all $n!$ permutations.

(b) Show that $P(R_n = r) = 1/n$ and that $R_1, R_2, \ldots$ are independent.

(c) Let $A_n = \{\text{record at time } n\} = \{\omega : X_n(\omega) > \max_{1\le k\le n-1} X_k(\omega)\}$ and show that $A_1, A_2, \ldots$ are independent and compute $P(A_n)$.

(d) Let N_n be the number of records up to time n and show that $N_n/\log n \to 1$ in probability.

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