SF3940– PROBABILITY THEORY

SPRING 2016

HOMEWORK 5

DUE APRIL 4, 2016

You must be able to explain the following concepts:

- Hewitt-Savage 0-1 law, the recurrence dichotomy, recurrence and transience of simple random walk on $\mathbb{Z}^d$, the Chung-Fuchs recurrence criteria, the reflection principle for simple random walks, the arcsin laws.

Solve *three* of the following problems.

PROBLEM 1. Show that if the random walk (S_n) is recurrent, then so is the random walk (S_{nk}) for each $k = 1, 2, \dots$.

PROBLEM 2. Let (S_n) be a random walk in $\mathbb{R}$ with increments from a symmetric non degenerate distribution with bounded support. Show that (S_n) is recurrent.

PROBLEM 3. Let $X_1, X_2, \dots$ be iid on $\mathbb{R}$ and $S_0 = 0$, $S_n = X_1 + \dots + X_n$, $n \geq 1$. Put $\tau_0 = 0$ and

$$\tau_k = \inf\{n > \tau_{k-1} : X_{\tau_{k-1}+1} + \dots + X_{\tau_{k-1}+n} > 0\}, \quad k \geq 1.$$

(a) Show that $P\{\tau_1 < \infty\} < 1 \Rightarrow P\{\sup_n S_n < \infty\} = 1$.
(b) Show that $P\{\tau_1 < \infty\} = 1 \Rightarrow P\{\sup_n S_n = \infty\} = 1$.
(c) Put $\sigma = \inf\{n : S_n < 0\}$ and show that the four combinations of $P\{\tau_1 < \infty\} < 1$ or $= 1$, and $P\{\sigma < \infty\} < 1$ or $= 1$ corresponds to the four possibilities $S_n = 0$ for all n, $S_n \to \infty$, $S_n \to -\infty$, and $-\infty = \liminf S_n < \limsup_n S_n = \infty$.

PROBLEM 4. Consider the previous exercise and let $\bar{\sigma} = \inf\{n : S_n \leq 0\}$. Show that

$$E\tau_1 = \frac{1}{P\{\bar{\sigma} = \infty\}}$$

PROBLEM 5. (a) Let (S_n) be a random walk based on the symmetric α-stable distribution on $\mathbb{R}$ with characteristic function $e^{-|t|^\alpha}$. Show that (S_n) is recurrent for $\alpha \geq 1$ and transient for $\alpha < 1$.

(b) Let (S_n) be a random walk based on the symmetric α-stable distribution on $\mathbb{R}^2$ with characteristic function $e^{-|t|^\alpha}$. Show that (S_n) is recurrent for $\alpha = 2$ and transient for $\alpha < 2$.

PROBLEM 6. For any symmetric, recurrent random walk on $\mathbb{Z}^d$, show that the expected number of visits to an accessible state $k \neq 0$ before return to the origin equals 1. (Hint: Compute the distribution, assuming probability p for return before visit to k.)

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