Introduction to game theory
LECTURE 1

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1 What is game theory?

A mathematically formalized theory of strategic interaction between

— countries at war and peace, in federations and international negotiations

— political candidates and parties competing for power

— firms in markets, owners and managers, employers and trade-unions

— members of communities with a common pool of resources

— family members and generations who care about each other's well-being

— animals within the same species, from different species, plants, cells

— agents in networks: computers, cell phones, vehicles in traffic systems

2 A brief history of game theory

- Emile Borel (1920s): Small zero-sum games
- John von Neumann (1928): the Maxmin Theorem
- von Neumann and Oskar Morgenstern (1944): Games and Economic Behavior
- John Nash (1950): Non-cooperative equilibrium ["A Beautiful Mind"]
- Thomas Schelling (1960s-): Strategic commitment, peace and war
- John Harsanyi (1960s): Incomplete information

- Reinhard Selten (1970s): Rationality as the limit of bounded rationality
- John Maynard Smith (1970s): Evolutionary stability
- Robert Aumann (1950s-): Long-run cooperation

3 Three simple examples

3.1 Prisoners' dilemma games

- Two fishermen, fishing in the same area

- Each fisherman can either fish modestly, M, or aggressively, A. The profits are

	M	A
M	$3, 3$	$1, 4$
A	$4, 1$	$2, 2$

- Both prefer (M, M), and both dislike (A, A)

- If each of them strives to maximize his or her profit, and they are both rational: (A, A)

- Competition leads to over-exploitation, not welfare maximum (What about Adam Smith's "invisible hand"?)

- Would monopoly be better?

3.2 Coordination games

- Two investors & two projects, *project* A and *project* B

- (A, A) gives higher expected profits to both investors than (B, B)

- Investment A has a positive externality on investment B

- Investor 1 chooses row, investor 2 column:

$$\begin{array}{ccc} & A & B \\ A & 5,5 & 0,4 \\ B & 4,0 & 3,3 \end{array}$$

- What are the Nash equilibria?

- If individuals were recurrently and (uniformly) randomly matched into pairs, in a large population, would there be any *"stable"* strategy?

- Pre-play communication: Suppose investor 2 suggests that you both invest in project A. Would that make investment alternative A more appealing?

- Note the belief indifference point: $\Pr(A) = 3/4$

- The notion of *risk dominance* (Harsanyi and Selten, 1988)

3.3 Partnership games

- Two partners in a business

- Each partner has to choose between "contribute" ("work") and "free-ride" ("shirk")

  - If both choose W: expected *gain to both*

  - If one chooses W and the other S: net *loss* to the first and *gain* to the second

  - If both choose S: expected *heavy loss to both*

	W	S
W	$3,3$	$-1,4$
S	$4,-1$	$-2,-2$

- This is **not** a Prisoners' Dilemma: S does not dominate W

- What are the Nash equilibria?

- If individuals are recurrently and randomly matched to pairs, in a large population, would there be any *evolutionarily stable* strategy?

$$\Pr(W) = \Pr(S) = \frac{1}{2}$$

4 Discussion

- Game theory as a paradigm for understanding strategic interaction

  in economics

  in political science

  in psychology and sociology

  in biology

  in computer science

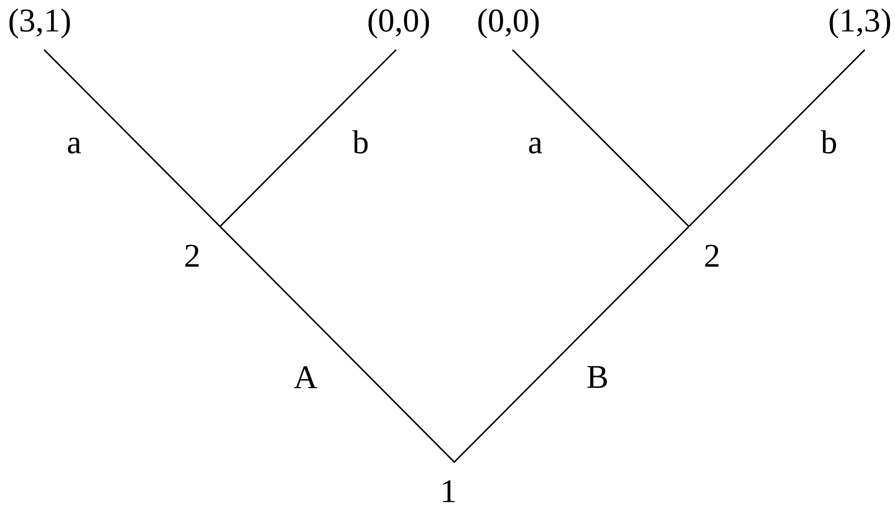
- Positive versus normative analysis

- Quantitative versus qualitative analysis

- Solution concepts: dominance, rationalizability, Nash equilibrium, subgame-perfect equilibrium, sequential equilibrium, perfect equilibrium, proper equilibrium, essential equilibrium, strategically stable sets, sets closed under rational behavior, evolutionary stability, equilibrium evolutionary stability...

- A game as a mathematical object: the *normal* (or *strategic*) form and the *extensive* form, on which we apply solution concepts

5 Informally about the extensive form

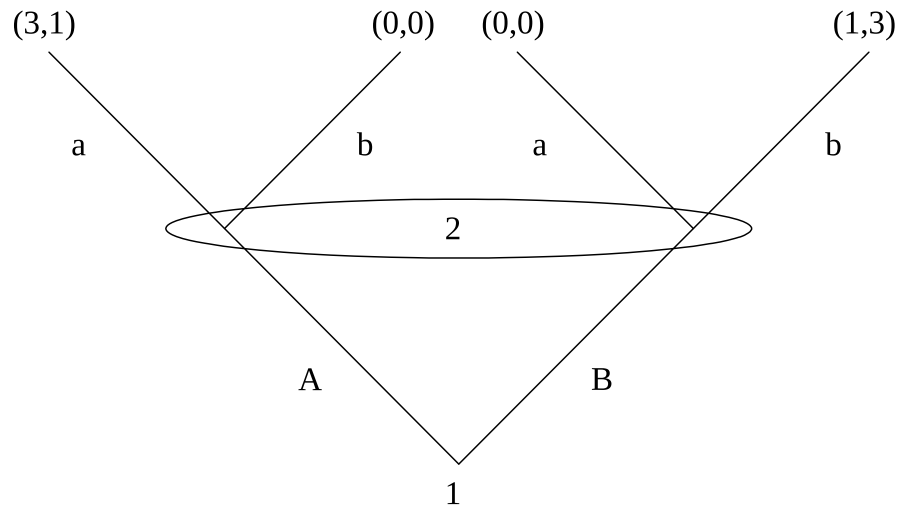


Game 1

- Four possible *plays*

- *Perfect*-information games vs. games of *imperfect* information

- Suppose that player 2 is *not* informed about 1's move:



Game 2

- In this game, player 2 cannot condition his choice on 1's action.

- Pure strategies in Game 1: $S_1 = \{A, B\}$, $S_2 = \{aa, ab, ba, bb\}$

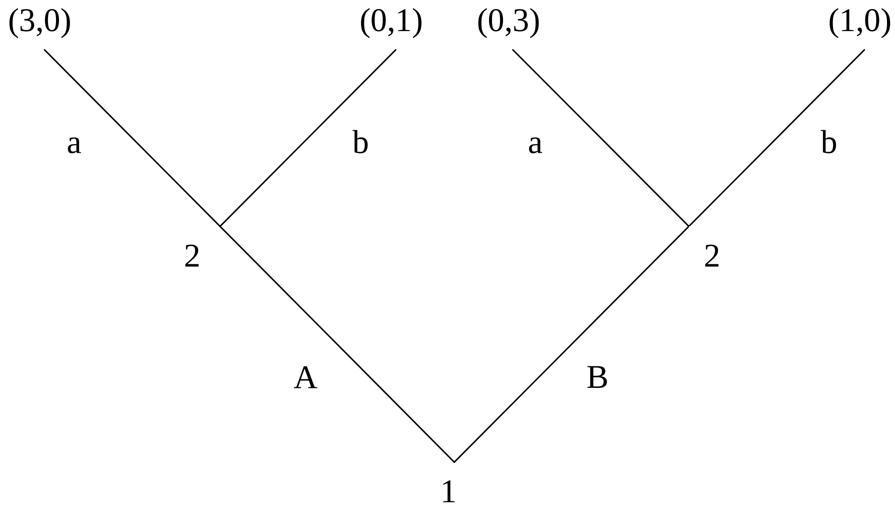
- Pure strategies in Game 2: $S_1' = \{A, B\}$, $S_2' = \{a, b\}$

- What should player 1 reasonably expect about 2's move in Game 1?

- *Backward induction*

- First-mover advantage

- Are there games with a *second*-mover advantage?



Game 3

6 Informally about the normal form

$$\begin{array}{llcccc} & & aa & ab & ba & bb \\ \text{Game 1:} & A & (3,1) & (3,1) & (0,0) & (0,0) \\ & B & (0,0) & (1,3) & (0,0) & (1,3) \end{array}$$

$$\begin{array}{llcc} & & a & b \\ \text{Game 2:} & A & (3,1) & (0,0) \\ & B & (0,0) & (1,3) \end{array}$$

- Pure and mixed strategies.

- Payoffs interpreted as the players' "utilities," both in the EF and in the NF.

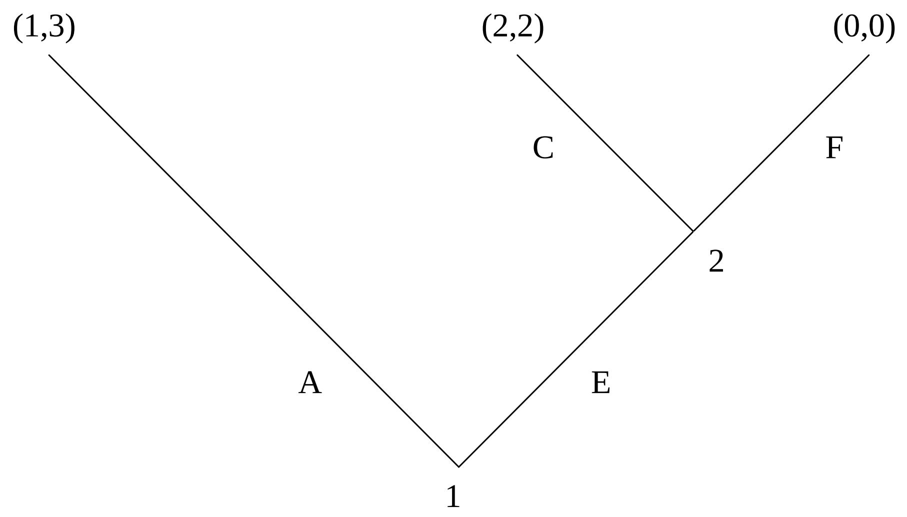
- Given the *normal*-form representation of Game 1, what is our prediction that player 1 will do?

- *Strictly and weakly dominated* strategies

- *Nash equilibrium:* a strategy profile such that if you expect the others to play according to it, then you cannot increase your own payoff by changing your own strategy [John Nash: "A Beautiful Mind", Economics Nobel memorial prize 1994]

7 Extensive forms with the same normal form

An *entry-deterrence* game: A potential entrant (player 1) into a monopolist's (player 2) market



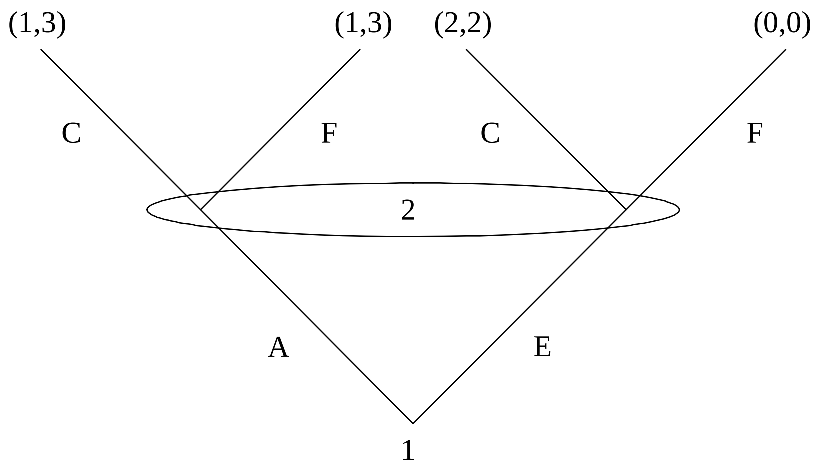
Game 4

- Its normal form:

$$\begin{array}{ccc} & C & F \\ A & 1,3 & 1,3 \\ E & 2,2 & 0,0 \end{array}$$

Two pure-strategy NE in this game: (A, F) and (E, C), but only the latter satisfies backward induction.

- Another extensive form game with the same normal form:



Game 5

- If players are rational, should the two extensive forms be deemed strategically equivalent?

- What if the players are boundedly rational?

8 Preferences and payoff functions

- A set X of *alternatives* $x, y, z...$

- Preferences as *binary relations* $\succcurlyeq$ on X: $x \succcurlyeq y$

  * *Transitivity*: if $x \succcurlyeq y$ and $y \succcurlyeq z$, then $x \succcurlyeq z$

  * *Completeness*: either $x \succcurlyeq y$ or $y \succcurlyeq x$ or both

- Indifference $x \sim y$ and strict preference $x \succ y$

Definition 8.1 *Let $\succcurlyeq$ be a binary relation on a set X. A* **utility function** *for $\succcurlyeq$ is a function $u : X \rightarrow \mathbb{R}$ such that $u(x) \geq u(y)$ iff $x \succcurlyeq y$.*

- Completeness and transitivity are *necessary* for such numerical representation

- If u is a utility function, then so is $v = h \circ u$ for any strictly increasing function $h : \mathbb{R} \to \mathbb{R}$: $u(x) \geq u(y) \Leftrightarrow h[u(x)] \geq h[u(y)] \Leftrightarrow v(x) \geq v(y)$.

In games:

1. For each player, a real-valued *function* over the set of possible *plays* of the game, the player's *Bernoulli function*.

2. For each player, a real-valued function over the set of strategy profiles of the game, the player's *payoff function*

8.1 Social preferences

- Most humans do not only care about their own material well-being but also about that of others

- The caring can be positive (*altruism)* or negative (*spitefulness).* People may like *fairness*, may have a desire to do as others' do (*conformity*, *social norms*), seek others' *esteem*, avoid *shame* or *guilt* etc.

- Consider again a *prisoners' dilemma*, but now with *monetary* payoffs (a so-called *game protocol*)

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$2, 2$

- D strictly dominates C, for each player, *in terms of monetary gains*

- $\Rightarrow (D, D)$ played *if both players are selfish*

- Suppose that each player cares about the other's monetary gain:

$$\begin{array}{ccc} & C & D \\ C & 3+3a, 3+3b & 4a, 4 \\ D & 4, 4b & 2+2a, 2+2b \end{array}$$

for some $a, b \in \mathbb{R}$

- If $a = b = 1/2$ (altruism of degree $1/2$):

$$\begin{array}{ccc} & C & D \\ C & 4.5, 4.5 & 2, 4 \\ D & 4, 2 & 3, 3 \end{array},$$

- Now both (C, C) and (D, D) are Nash equilibria: *A coordination game!*

- Hence: altruistic and rational individuals may (but need not) cooperate in a prisoners' dilemma protocol.

9 Some mathematics

9.1 Notation and tools

1. **Useful sets**: $\mathbb{N}$ the positive integers, $\mathbb{R}$ the reals, $\mathbb{R}_+$ the non-negative reals, $\mathbb{Q} \subset \mathbb{R}$ the rationals

2. **Definition**: *Injections* $f : X \to Y$:

$$x \neq x' \;\Rightarrow\; f(x) \neq f\left(x'\right)$$

3. **Definition:** A set X is *countable* if $\exists$ injection $f : X \to \mathbb{N}$

4. **Definitions:** *open*, *closed* and bounded sets $X \subset \mathbb{R}^n$, the *interior* and *closure* of sets $X \subset \mathbb{R}^n$

5. **Definition**: *upper-contour* sets for a function $f : X \to \mathbb{R}$

$$\{x \in X : f(x) \geq \alpha\}$$

6. **Definition**: *convex* sets $X \subset \mathbb{R}^n$

$$x, y \in X \Rightarrow \lambda x + (1 - \lambda) y \in X \quad \forall \lambda \in [0, 1]$$

7. **Definition**: A function $f : \mathbb{R}^n \to \mathbb{R}$ is *quasi-concave* if all its upper contour-sets are convex

8. **Definition**: Given a function $f : X \to \mathbb{R}$

$$\arg\max_{x \in X} f(x) := \{x^* \in X : f(x^*) \geq f(x) \ \forall x \in X\}$$

9. **Some useful results:**

Theorem 9.1 (Weierstrass' Maxium Theorem) *If $X \subset \mathbb{R}^n$ is non-empty and compact, and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, then* $\arg\max_{x \in X} f(x)$ *is non-empty and compact.*

Proposition 9.2 *If $X \subset \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}$ quasi-concave, then* $\arg\max_{x \in X} f(x)$ *is convex.*

9.2 Correspondences

A *correspondence* φ from a set X to a set Y, written $\varphi : X \rightrightarrows Y$, is a function that assigns a non-empty *set* $\varphi(x) \subset Y$ to each $x \in X$. (Hence, a non-empty valued *function* from X to 2^Y.)

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$:

Definition 9.1 *A correspondence $\varphi : X \rightrightarrows Y$ is* **upper hemi-continuous (u.h.c.)** *at $x \in X$ if for every open set B containing $\varphi(x)$ there exists an open set A such that $x \in A$ and $\varphi(x') \subset B \; \forall x' \in A \cap X$.*

Definition 9.2 *A correspondence $\varphi : X \rightrightarrows Y$ is* **lower hemi-continuous (l.h.c.)** *at $x \in X$ if for every open set B such that $\varphi(x) \cap B \neq \varnothing$ there exists a neighborhood A of x such that $\varphi(x') \cap B \neq \varnothing \; \forall x' \in A \cap X$.*

Definition 9.3 *A correspondence* $\varphi : X \rightrightarrows Y$ *is* **continuous** *at* $x \in X$ *if it is both u.h.c. and l.h.c. at* x.

Definition 9.4 *A correspondence is u.h.c. (l.h.c., continuous) if it is u.h.c. (l.h.c., continuous) at each point in its domain.*

9.3 Fixed-Point Theorems

Theorem 9.3 (Brouwer's Fixed-Point Theorem) *If* $X \subset \mathbb{R}^n$ *is non-empty, compact and convex, and* $f : X \to X$ *is continuous, then there exists at least one* $x^* \in X$ *such that* $x^* = f(x^*)$.

Theorem 9.4 (Kakutani's Fixed-Point Theorem) *If* $X \subset \mathbb{R}^n$ *is non-empty, compact and convex, and* $\varphi : X \rightrightarrows X$ *is convex-valued, closed-valued and u.h.c., then there exists at least one* $x^* \in X$ *such that* $x^* \in \varphi(x^*)$.

9.4 Berge's Maximum Theorem

Consider

$$\max_{x \in \gamma(a)} f(x, a)$$

where $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ is continuous and $\gamma : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ is compact-valued. Let

$$v(a) = \max_{x \in \gamma(a)} f(x, a) \quad \text{and} \quad \xi(a) = \arg \max_{x \in \gamma(a)} f(x, a)$$

Here γ is called the *constraint correspondence*, v the *value function* and ξ the *solution correspondence* (a selection from γ: $\xi(a) \subset \gamma(a)\ \forall a$.)

Theorem 9.5 (Berge's Maximum Theorem) *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ and $\gamma : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ are continuous. Then $\xi : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ is u.h.c. and compact-valued, and $v : \mathbb{R}^k \to \mathbb{R}$ is continuous.*