Introduction to game theory

LECTURE 2

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Two topics today:

1. *Existence of Nash equilibria* (Lecture notes Chapter 10 and Appendix A)

2. *Relations between equilibrium and rationality* (Lecture notes Chapter 1)

1 Existence of Nash equilibria

- Broad classes of normal-form games, results that have von Neumann's (1928) minimax theorem and Nash's (1950a,b) existence results as special cases.

Definition 1.1 *A* **normal-form game** *is a triplet* $G = (N, S, \pi)$, *where*

(i) N *is the set of* **players**

(ii) $S = \times_{i \in N} S_i$ *is the set of* **strategy profiles** $s = (s_i)_{i \in N}$ *with* S_i *denoting the* **strategy set** *of player* i

(iii) $\pi : S \to \mathbb{R}^n$ *is the* **combined payoff function**, *where* $\pi_i(s) \in \mathbb{R}$ *is the payoff to player* i *when strategy profile* s *is played.*

- Notation: for any strategy profile $s \in S$, player $i \in N$ and strategy $s_i' \in S_i$, write $\left(s_i', s_{-i}\right)$ for the strategy profile in which s_i has been replaced by s_i'

- Notation: for any strategy profile $s \in S$ and player $i \in N$, write

$$\begin{aligned} \beta_i(s) &= \arg\max_{s_i' \in S_i} \pi_i\left(s_i', s_{-i}\right) \\ &= \left\{s_i' \in S_i : \pi_i\left(s_i', s_{-i}\right) \geq \pi_i\left(s_i'', s_{-i}\right) \quad \forall s_i'' \in S_i\right\} \end{aligned}$$

- This defines player $i's$ (possibly empty-valued) best-reply correspondence $\beta_i : S \rightrightarrows S_i$

- Write $\beta(s) = \times_{i \in N} \beta_i(s)$

- This defines the (possibly empty-valued) best-reply correspondence $\beta : S \rightrightarrows S$ of the game G

Definition 1.2 *A strategy profile* $s^* \in S$ *is a Nash equilibrium if* $s^* \in \beta(s^*)$

- Mixed strategies and *the mixed-strategy extension* $\tilde{G} = (N, M, \tilde{\pi})$ of G.

1.1 Games in Euclidean spaces

- By a *Euclidean game* we mean a game in which (a) the set N is finite, (b) each strategy set S_i is a subset of $\mathbb{R}^{m_i}$ for some $m_i \in \mathbb{N}$.

Proposition 1.1 *Let $G = (N, S, \pi)$ be a Euclidean game in which each strategy set S_i is non-empty, compact and convex, each payoff function $\pi_i : S \to \mathbb{R}$ is continuous, and, moreover, quasi-concave in $s_i \in S_i$. Then G has at least one Nash equilibrium.*

Definition 1.3 *A function $f : X \to \mathbb{R}$, where X is a convex subset of some linear vector space, is* **quasi-concave** *if all its upper-contour sets*

$$X_a = \{x \in X : f(x) \geq a\}$$

are convex.

Proof of the proposition in class:

1. Weierstrass' Maximum Theorem

2. Continuity properties of correspondences (Kuratovski, Berge)

3. Berge's Maxiumum Theorem

4. Kakutani's Fixed-Point Theorem

- Clearly Nash's (1950) existence result is a special case. Proof in class.

- Consider now Euclidean games $G = (N, S, \pi)$ in which all strategy sets are compact and all payoff functions π_i continuous

- Its *mixed-strategy extension* is the normal-form game $\tilde{G}\,(N, M, \tilde{\pi})$ in which

  (a) For each player i, $M_i = \Delta\,(S_i)$ is the set of Borel probability measures μ_i on S_i

  (b) $M = \times_{i=1}^{n} M_i$.

  (c) $\tilde{\pi}_i : M \to \mathbb{R}$ is obtained by taking the mathematical expectations of the player's payoff function $\pi_i : S_i \to \mathbb{R}$, with respect to the product measure μ on S that arise from each mixed-strategy profile, $\mu = (\mu_1, ..., \mu_n)$:

$$\tilde{\pi}_i\,(\mu) = \int_{S_1} ... \int_{S_n} \pi_i\,(s_1, ..., s_n)\, d\mu_1\,(s_1) \cdot ... \cdot d\mu_n\,(s_n)$$

Special case: G finite.

- The following result is a special case of a theorem due to Glicksberg (1952):

Proposition 1.2 *Let $G = (N, S, \pi)$ be a Euclidean game in which each strategy set S_i is non-empty and compact, and where each payoff function $\pi_i : S \to \mathbb{R}$ is continuous. Then its mixed-strategy extension, the game $\tilde{G} = (N, M, \tilde{\pi})$, has at least one Nash equilibrium.*

Proof idea: View M as a convex linear topological vector space and generalize Kakutani.

1.2 Games in topological vector spaces

Definition 1.4 *A* **topological vector space** *is a linear vector space V endowed with a topology such that vector addition (viewed as a function from $V \times V$ to V) and scalar multiplication (viewed as a mapping from $\mathbb{R} \times V$ to V) are continuous.*

Example: Normed linear vector spaces.

- Consider games $G = (N, S, \pi)$ in which each strategy set S_i is a compact subset of some topological vector space, and where each payoff function is bounded

Definition 1.5 *The graph of the combined payoff function $\pi : S \to \mathbb{R}^n$, or* **the payoff graph** *for short, is the set*

$$\Pi = \{(s, v) \in S \times \mathbb{R}^n : \pi(s) = v\}$$

- Note: If all strategy sets are compact and all payoff functions continuous, then the payoff graph is closed, while this need not be the case if some or all payoff functions are discontinuous

- Let $\bar{\Pi}$ be the closure of the payoff graph.

Definition 1.6 (Reny) *A vector-space game* $G = (N, S, \pi)$ *is* **payoff secure** *if, for each strategy profile* $s \in S$ *and* $\varepsilon > 0$, *there exists a pure strategy* $\hat{s}_i \in S_i$ *for each player* i *such that* $\pi_i\left(\hat{s}_i, s'_{-i}\right) \geq \pi_i(s) - \varepsilon$ *for all* $s' \in S$ *in some neighborhood of* s.

Definition 1.7 (Reny) *A vector-space game* $G = (N, S, \pi)$ *is* **reciprocally upper semi-continuous** *(reciprocally u.s.c.) if, for all* $s \in S$ *and* $v \in \mathbb{R}^n$:

$$(s, v) \in \bar{\Pi} \text{ and } \pi(s) \leq v \;\Rightarrow\; \pi(s) = v$$

- This condition admits all games with continuous payoff functions

- The condition requires that if one player's payoff discontinuously jumps up, then some other player's payoff simultaneously has to jump down.

- The condition is met if *the payoff sum* is continuous (as it is in many price-competition games).

Theorem 1.3 (Reny) *Let $G = (N, S, \pi)$ be a vector-space game in which each strategy set S_i is non-empty, compact and convex, and where each payoff function $\pi_i : S \to \mathbb{R}$ is bounded and quasi-concave in $s_i \in S_i$. If G is payoff-secure and reciprocally u.s.c., then G has at least one Nash equilibrium.*

- This result has Proposition 1.1 as an immediate corollary.

- It applies to some discontinuous games that arise in models of price competition and auctions.

- Consider now the mixed-strategy extension, $\tilde{G} = (N, M, \tilde{\pi})$, of a vector-space game $G = (N, S, \pi)$, where each set M_i is the set of regular Borel measures on S_i.

Proposition 1.4 (Reny) *Let $G = (N, S, \pi)$ be a vector-space game in which each strategy set S_i is non-empty, Hausdorff, compact and convex, and where each payoff function $\pi_i : S \to \mathbb{R}$ is bounded and Borel measurable, and, moreover, quasi-concave in s_i. If $\tilde{G} = (N, M, \tilde{\pi})$ is payoff-secure and reciprocally u.s.c., then $\tilde{G}$ has at least one Nash equilibrium.*

Remark 1.1 *Reciprocal upper semi-continuity of $\tilde{G}$ implies that of G. However, reciprocal upper semi-continuity of G does not imply that of G, and payoff security of G neither implies nor is implied by payoff security of $\tilde{G}$.*

2 Equilibrium and rationality

2.1 The rationalistic interpretation

1. All players are *rational* in the sense of Savage (1954): players only use strategies that are optimal under *some* probabilistic belief about the other's strategy choices

   - and assume statistical independence between players' choices

2. Each player *knows* G

But... these assumption do not imply NE in general

Assume more!

- Common knowledge (Lewis 1969, Aumann 1974) of the game and of all players' (Savage) rationality

  - they all know G, know that all know G, know that all know that all know G etc.

  - they all know that all players are rational, that all players know that all players are rational etc.

Still does not imply NE...

Example 2.1 *("Battle of the Sexes")*

$$\begin{array}{ccc} & a & b \\ A & 3,1 & 0,0 \\ B & 0,0 & 1,3 \end{array}.$$

Example 2.2 *("Matching Pennies")*

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Example 2.3 *(unique and strict NE)*

	L	C	R
T	$7,0$	$2,5$	$0,7$
M	$5,2$	$3,3$	$5,2$
B	$0,7$	$2,5$	$7,0$

- Epistemic foundations of NE: Aumann and Brandenburger (1995)

- The above assumptions lead to **rationalizability**, not **equilibrium**

2.2 The mass-action interpretation

“It is unnecessary to assume that the participants in a game have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.

To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure strategy is employed by the 'average member' of the appropriate population.

Since there is to be no collaboration between individuals playing in different positions of the game, the probability that a particular n-tuple of pure strategies will be employed in playing of the game should be the product of the probabilities indicating the chance of each of the n pure strategies to be employed in a random playing.

... Thus the assumptions we made in this 'mass action' interpretation led to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point." (*John Nash's PhD. thesis*)

But... these assumption do not imply NE in general!

- Evolutionary game theory (replication of behaviors in large populations)

3 Next lecture

Two topics also the next two lectures (Monday and Wednesday next week):

1. *Finite extensive- and normal-form games* (Lecture notes Chapter 3)

2. *Solution concepts for finite normal-form games* (Lecture notes Chapter 4 and/or EGT book Chapter 1)

THE END