1 Extensive-form games


Definition 1.1 A finite extensive-form game is a 9-tuple

\[ \Gamma = (N, A, \psi, \mathcal{P}, \mathcal{I}, C, p, r, v) \] where:

1. \( N = \{1, \ldots, n\} \) the set of personal players

2. \( A \) the set of nodes

\[ a_0 \in A \] the root
3. $\psi : A \setminus \{a_0\} \rightarrow A$ the \textit{predecessor function}

notation: $a < a' \iff a = \psi(a')$

$A_\omega \subset A$ the \textit{terminal} nodes

$T$ the set of \textit{plays} $\tau$

4. $P = \{P_0, P_1, \ldots, P_n\}$ the \textit{player partitioning} of non-terminal nodes, allowing for empty sets

(If $P_i = \emptyset$ then $i$ is called a \textit{null player}.)

5. $I = \bigcup_{i \in N} I_i$, where each $I_i$ is the \textit{information partitioning} of $P_i \subset A$ into (non-empty) \textit{information sets} $I \in I_i$. Two regularity conditions:

(5a) Each play intersects every information set at most once

(5b) All nodes in an info set have the same number of outgoing branches
6. \( C = \{C_I : I \in \mathcal{I}\} \), where each \( C_I \) is the choice partitioning of outgoing branches at \( I \)

   notation: \( c < a \)

7. \( p \) the probabilities of “nature’s” random moves at nodes \( a \in P_0 \)

8. \( r : T \to D \) the result function (or outcome function), assigning material consequences to plays

9. \( v : T \to \mathbb{R}^n \) the combined Bernoulli function, assigning Bernoulli values, \( v_i(\tau) \in \mathbb{R} \), to each play \( \tau \) and player \( i \).

   These values represent how “good” or “bad” the plays are for the player, and may depend on all details of \( N, A, \psi, \mathcal{P}, \mathcal{I}, C, p, r \).

   • Note that here: \( T \leftrightarrow A_\omega \) (but not in infinite-horizon games)
Distinction between:

- $\Phi = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p)$ the game form

- $\Psi = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, r)$ the game protocol (or mechanism)

- $\Gamma = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, r, v)$ the game

Later on, when we work with solution concepts, the function $r$ will not matter (explicitly), only the function $v$. 
Is this an extensive form?
What about this one?
2 Game theory is not consequentialistic

- Preferences over plays \( \neq \) preferences over results (consequences)

Example 2.1 Let the numbers be monetary gains (say euros):
3 Subgames

- The follower set

\[ F(a) = \{ a' \in A : a \leq a' \} \]

- Subroots are nodes \( a \) for which:

\[ F(a) \cap I \neq \emptyset \Rightarrow I \subset F(a). \]

**Definition 3.1** A subgame of \( \Gamma \) is the tree starting at a subroot \( a \), endowed with the same partitionings etc. and denoted \( \Gamma_a \) (in particular, \( \Gamma_{a_0} = \Gamma \) is a subgame)
4 Strategies, realization probabilities and payoff functions

4.1 Pure strategies

Definition 4.1 A pure strategy $s_i$ for a player $i$ is a function that assigns a choice $c \in C_I$ to each information set $I \in I_i$ of the player.

- Note that a pure strategy is more than what people usually think...

- Pure-strategy profiles $s = (s_1, ..., s_n) \in S = \times_{i \in N}S_i$
• Realization probabilities for plays $\tau \in T$: $\rho(\tau, s)$ is the probability for $\tau$ under $s \in S$

**Definition 4.2**  *The pure-strategy payoff function* $\pi_i : S \rightarrow \mathbb{R}$ for player $i$ is defined by

$$\pi_i(s) = \sum_{\tau \in T} \rho(\tau, s) v_i(\tau)$$
4.2 Mixed strategies

Definition 4.3 A mixed strategy $x_i$ for player $i$ is a probability distribution over $i$'s set of pure strategies.

- As if each player randomizes before starting to play
- Notation: $x_i \in X_i = \Delta(S_i)$

- Mixed-strategy profiles

$$x = (x_1, ..., x_n) \in X = \boxtimes(S) = \times_i \Delta(S_i)$$

- Realization probabilities:

$$\tilde{\rho}(\tau, x) = \sum_{s \in S} \left[ \prod_{j \in N} x_j(s_j) \right] \rho(\tau, s)$$
Definition 4.4  *The mixed-strategy payoff function* $\tilde{\pi}_i : \square(S) \to \mathbb{R}$ *for player* $i$ *is defined by*

$$\tilde{\pi}_i(x) = \sum_{\tau \in T} \tilde{\rho}(\tau, x) v_i(\tau)$$
• Polynomial functions

Example 4.1

\[
\left\{
\begin{align*}
\tilde{\pi}_1 (x) &= x_{11} + 2x_{12}x_{21} \\
\tilde{\pi}_2 (x) &= 3x_{11} + 2x_{12}x_{21}
\end{align*}
\right.
\]
4.3 Behavior strategies

- *Local strategies*: statistically independent randomizations over choice sets,

\[ y_{iI} \in Y_{iI} = \Delta(C_I) \]

**Definition 4.5** A behavior strategy \( y_i \) for player \( i \) is a function that assigns a local strategy to each information set \( I \in \mathcal{I}_i \) of the player

- As if players randomize as play proceeds

- Notation: \( y_i \in Y_i = \times_{I \in \mathcal{I}_i} Y_{iI} \)

- Behavior-strategy profiles: \( y \in Y = \times_{i \in N} Y_i \)
• Realization probabilities: \( \hat{\rho}(\tau, y) = \) the product of all choice probabilities along \( \tau \)

**Definition 4.6** The behavior-strategy payoff function \( \hat{\pi}_i \) of player \( i \) is defined by

\[
\hat{\pi}_i (y) = \sum_{\tau \in T} \hat{\rho}(\tau, y) v_i(\tau)
\]
4.4 Outcome and path

- Terminology for pure, mixed and behavior strategy profiles:

**Definition 4.7** Outcome of strategy profile = induced probability distribution over plays

**Definition 4.8** Path of strategy profile = the set of plays assigned positive probabilities = the support of the outcome.

- Also applied to nodes and information sets “on and off the path”.
5 Perfect recall and Kuhn’s theorem

- Mixed strategies: “global randomizations” performed at the beginning of the play of the game

- Behavior strategies: “local randomizations” performed during the course of play of the game

- Equivalence in terms of realization probabilities?
Definition 5.1 (Kuhn 1950,1953) An extensive form $\Phi$ has perfect recall if

$$c < a \Leftrightarrow c < a'$$

for each player $i \in N$, pair of information sets $I, J \in I_i$, choice $c \in C_I$ and nodes $a, a' \in J$.

- Note: An extensive form has perfect recall if each player has only one information set.

- Note: Bernoulli values and payoffs are irrelevant for this definition.

Informally:

Theorem 5.1 ("Kuhn’s Theorem") If $\Phi$ has perfect recall, then, for each mixed strategy, $\exists$ a realization-equivalent behavior strategy.
5.1 Behavior-strategy mixtures

To state this more exactly: Consider a player \( i \) in a finite extensive form \( \Phi \).

Definition 5.2 A (behavior-strategy) mixture, \( w_i \), is a finite-support randomization over the player’s set of behavior strategies: \( w_i \in W_i \), where \( W_i \) is the set of probability vectors \( w_i = (w_i(y_i^1),...,w_i(y_i^k)) \) for some \( k \in \mathbb{N} \) and \( y_i^1,...,y_i^k \in Y_i \).

- Every behavior strategy \( y_i \in Y_i \) can be viewed as a (degenerate) behavior-strategy mixture, the mixture \( w_i \) that assigns unit probability to \( y_i \).

- Every mixed strategy \( x_i \in X_i \) can be viewed as the mixture \( w_i \) that assigns probability \( x_{ih} \in [0,1] \) to the (degenerate) behavior strategy \( y_i^h \) that assigns unit probability to the choices made under pure strategy \( h \in S_i \).
Definition 5.3 A mixture \( w'_i \in W_i \) is realization equivalent with a mixture \( w_i \in W_i \) if the realization probabilities under the profile \( (w'_i, w''_i) \) are identical with those under \( (w_i, w''_i) \), for all profiles \( w'' \in \times_{j=1}^n W_j \).

Theorem 5.2 (Kuhn 1950, Selten 1975) Consider a player \( i \) in a finite extensive form \( \Phi \) with perfect recall. For each behavior-strategy mixture \( w_i \in W_i \) there exists a realization-equivalent mixture \( w'_i \in W_i \) that assigns unit probability to a behavior strategy \( y_i \in Y_i \).

Rough proof sketch:

1. Consider those of \( i \)'s information sets \( I \) that are possible under \( w_i \) in the sense that \( I \) is on the path of \( (w_i, w''_i) \) for some \( w'' \in \times_{j=1}^n W_j \)

2. Note that conditional probabilities across nodes in an information set \( I \in \mathcal{I}_i \) do not depend on \( i \)'s own strategy
6 Normal-form games

- A normal-form game: a triplet $G = (N, S, \pi)$ where

  $N$ is the set of players

  $S = \times_{i \in N} S_i$ the set of strategy profiles $s = (s_i)_{i \in N}$, $S_i$ the strategy set of player $i$

  $\pi : S \rightarrow \mathbb{R}^n$ is the combined payoff function, $\pi_i(s) \in \mathbb{R}$ the payoff to player $i$ under $s$
Example 6.1 A firm offering a wage $w \in W = [0, 100]$ to a worker, who can accept or reject the offer. If accept ($y = 1$), then $v_1 = 100 - w$ (profit) and $v_2 = w$ (utility). If reject ($y = 0$), then $v_1 = v_2 = 0$. The normal form:

$S_1 = W = [0, 100]$ 

$S_2 = \{0, 1\}^W$; the set of functions $f : W \rightarrow \{0, 1\}$

$\pi_1 (w, f) = (100 - w) \cdot f (w)$

$\pi_2 (w, f) = w \cdot f (w)$
7 Five NF games associated with each EF game

- $G$ is called finite if both $N$ and $S$ are finite

- Five normal form for a given EF game $\Gamma$:
  1. The pure-strategy normal form $G = (N, S, \pi)$
  2. The mixed-strategy normal form $\tilde{G} = (N, X, \tilde{\pi})$
  3. The behavior-strategy normal form $\hat{G} = (N, Y, \hat{\pi})$
  4. The quasi-reduced normal form
  5. The reduced normal form
Example 7.1

**Game 9**

<table>
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<th>D</th>
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<td>2,1</td>
</tr>
<tr>
<td>AF</td>
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<td>2,1</td>
</tr>
<tr>
<td>BE</td>
<td>1,2</td>
<td>3,2</td>
</tr>
<tr>
<td>BF</td>
<td>1,2</td>
<td>0,0</td>
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</tbody>
</table>
Quasi-reduced (and reduced):

<table>
<thead>
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<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2,1</td>
<td>2,1</td>
</tr>
<tr>
<td>$BE$</td>
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<td>3,2</td>
</tr>
<tr>
<td>$BF$</td>
<td>1,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>
8 Thompson’s transformations

- Thompson (1952) studied four “strategically inessential” transformations of finite extensive-form games (see also Kohlberg and Mertens, 1986).

- Thompson showed that by successive application of these transformations, any finite extensive-form game can be rendered on the form of a simultaneous-move game.

- However, one of these transformations (called inflate-deflate) may result in a game without perfect recall.

- Elmes and Reny (1994) proved that one can dispense with that transformation if one of the other transformations is slightly modified.
• The three transformations are “add”, “coalesce” and “interchange”

1. “Add”, consists in adding a node to a player’s information set in such a way that the player’s choice will not affect any player’s payoff in case play would reach the added node. [Reconsider the entry-deterrence game in lecture 1]

2. “Coalesce” brings together two consecutive decision nodes, each being a singleton information set and belonging to the same player. [Example in class]

3. “Interchange” changes the order of moves between two players who are not informed of each others’ moves. [Reconsider Game 2 in lecture 1]
Figure 1:

**Theorem 8.1 (Elmes and Reny)**  If $\Gamma$ and $\Gamma'$ are extensive-form games with perfect recall and have the same quasi-reduced normal form, then there exists a finite sequence of games, $\Gamma_1, \ldots, \Gamma_k$, each with perfect recall, such that (a) $\Gamma_1 = \Gamma$ and $\Gamma_k = \Gamma'$ and (b) consecutive games in the sequence differ only by one of the transformations “add”, “coalesce” or “interchange”.

- Hence, if we, as analysts, deem the three transformations “strategically inessential” then we will prefer solution concepts that are invariant under these transformations, that is, that (by the above theorem) depend only on the quasi-reduced normal form. [See discussion in Kohlberg and Mertens, 1986]
9 Solution concepts

Now we are in a position to define and analyze different solution concepts for games

- Solution concepts for extensive-form games (complicated math)

- Solution concepts for normal-form concepts (easier math)

- Interpretations of solutions: (a) rationalistic, (b) evolutionary

Next topic:

Solution concepts for finite normal-form games.