EXTENSIVE AND NORMAL FORM GAMES

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February 9, 2010

1 Extensive-form games

- Kuhn (1950,1953), Selten (1975), Kreps and Wilson (1982), Weibull (2004)

Definition 1.1 *A finite* **extensive-form game** *is a 9-tuple*

$\Gamma = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, r, v)$ *where:*

1. $N = \{1, ..., n\}$ the set of *personal players*

2. A the set of *nodes*

   $a_0 \in A$ the *root*

3. $\psi : A \backslash \{a_0\} \rightarrow A$ the *predecessor function*

   notation: $a < a' \Leftrightarrow a = \psi(a')$

   $A_\omega \subset A$ the *terminal* nodes

   T the set of *plays* τ

4. $\mathcal{P} = \{P_0, P_1, ..., P_n\}$ the *player partitioning* of non-terminal nodes, allowing for empty sets

   (If $P_i = \varnothing$ then i is called a *null player*.)

5. $\mathcal{I} = \cup_{i \in N} \mathcal{I}_i$, where each $\mathcal{I}_i$ is the *information partitioning* of $P_i \subset A$ into (non-empty) *information sets* $I \in \mathcal{I}_i$. Two regularity conditions:

   (5a) Each play intersects every information set at most once

   (5b) All nodes in an info set have the same number of outgoing branches

6. $\mathcal{C} = \{C_I : I \in \mathcal{I}\}$, where each C_I is the *choice partitioning* of outgoing branches at I

   notation: $c < a$

7. p the probabilities of "nature's" random moves at nodes $a \in P_0$

8. $r : T \to D$ the *result function* (or *outcome function*), assigning material consequences to plays

9. $v : T \to \mathbb{R}^n$ the combined Bernoulli function, assigning Bernoulli values, $v_i(\tau) \in \mathbb{R}$, to each play τ and player i.

   These values represent how "good" or "bad" the plays are for the player, and may depend on all details of $N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, r$.

- Note that here: $T \leftrightarrow A_\omega$ (but not in infinite-horizon games)

Distinction between:

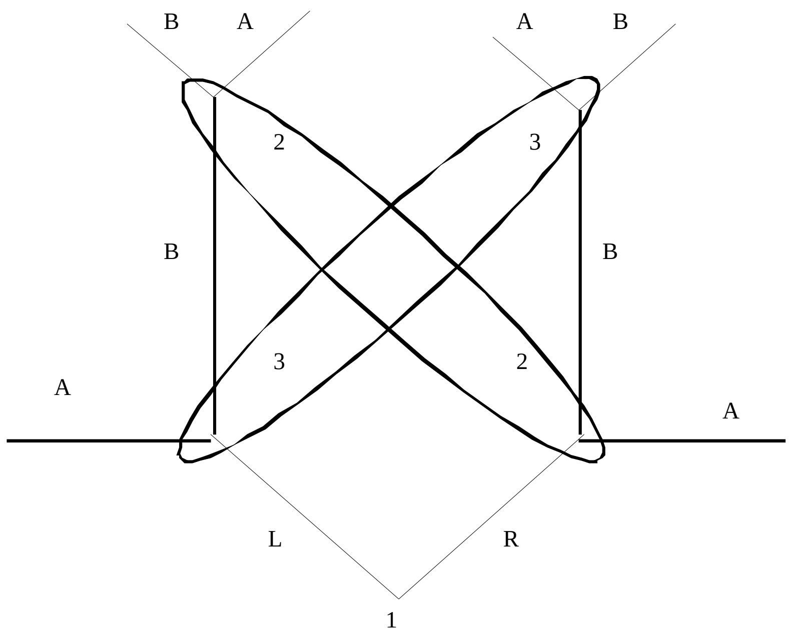
- $\Phi = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p)$ the **game form**

- $\Psi = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, r)$ the **game protocol (or mechanism)**

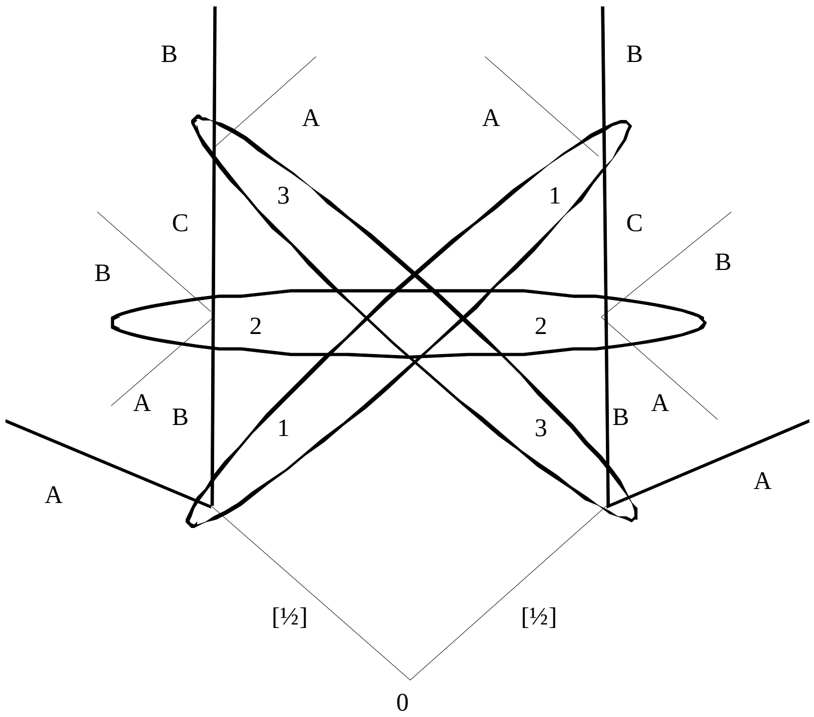
- $\Gamma = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, r, v)$ the **game**

Later on, when we work with solution concepts, the function r will not matter (explicitly), only the function v

Is this an extensive form?



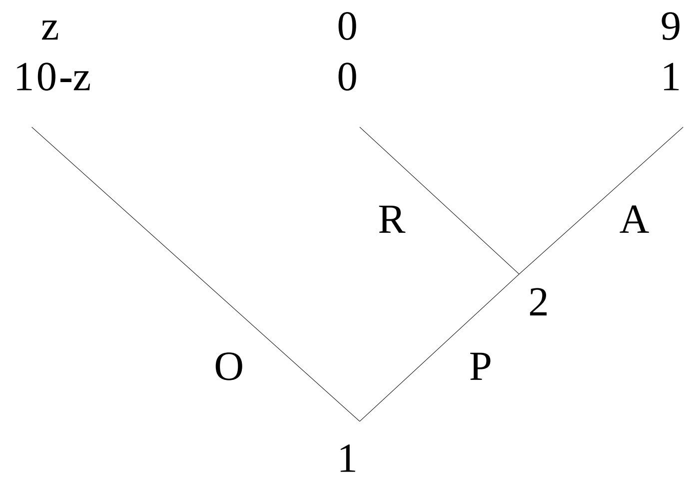
What about this one?



2 Game theory is not consequentialistic

- Preferences **over plays** $\neq$ preferences **over results (consequences)**

Example 2.1 *Let the numbers be monetary gains (say euros):*



3 Subgames

- The *follower set*

$$F(a) = \left\{a' \in A : a \leq a'\right\}$$

- *Subroots* are nodes a for which:

$$F(a) \cap I \neq \varnothing \Rightarrow I \subset F(a).$$

Definition 3.1 *A* **subgame** *of* Γ *is the tree starting at a subroot* a*, endowed with the same partitionings etc. and denoted* Γ_a *(in particular,* $\Gamma_{a_0} = \Gamma$ *is a subgame)*

4 Strategies, realization probabilities and payoff functions

4.1 Pure strategies

Definition 4.1 *A* **pure strategy** s_i *for a player* i *is a function that assigns a choice* $c \in C_I$ *to each information set* $I \in \mathcal{I}_i$ *of the player*

- Note that a pure strategy is more than what people usually think...

- Pure-strategy profiles $s = (s_1, ..., s_n) \in S = \times_{i \in N} S_i$

- Realization probabilities for plays $\tau \in T$: $\rho(\tau, s)$ is the probability for τ under $s \in S$

Definition 4.2 *The* **pure-strategy payoff function** $\pi_i : S \to \mathbb{R}$ *for player* i *is defined by*

$$\pi_i(s) = \sum_{\tau \in T} \rho(\tau, s)\, v_i(\tau)$$

4.2 Mixed strategies

Definition 4.3 *A* **mixed strategy** x_i *for player* i *is a probability distribution over* $i's$ *set of pure strategies.*

- As if each player randomizes before starting to play
- Notation: $x_i \in X_i = \Delta(S_i)$
- Mixed-strategy profiles

$$x = (x_1, ..., x_n) \in X = \square(S) = \times_i \Delta(S_i)$$

- Realization probabilities:

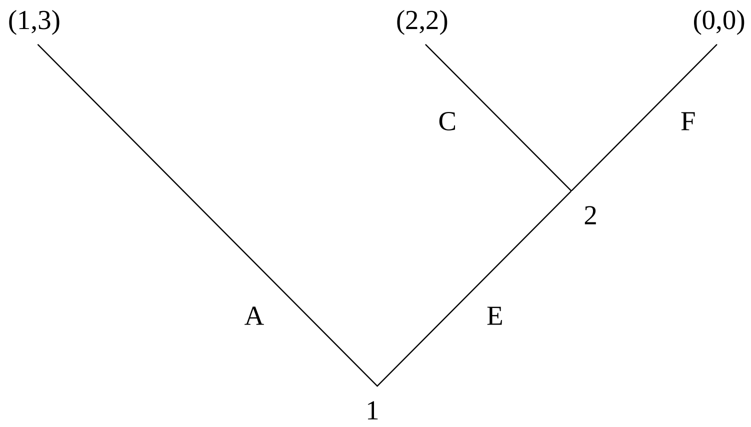
$$\tilde{\rho}(\tau, x) = \sum_{s \in S} \left[\Pi_{j \in N} x_j \left(s_j \right) \right] \rho(\tau, s)$$

Definition 4.4 *The* **mixed-strategy payoff function** $\tilde{\pi}_i : \square(S) \longrightarrow \mathbb{R}$ *for player* i *is defined by*

$$\tilde{\pi}_i(x) = \sum_{\tau \in T} \tilde{\rho}(\tau, x) \, v_i(\tau)$$

- Polynomial functions

Example 4.1



Game 4

$$\begin{cases} \tilde{\pi}_1(x) = x_{11} + 2x_{12}x_{21} \\ \tilde{\pi}_2(x) = 3x_{11} + 2x_{12}x_{21} \end{cases}$$

4.3 Behavior strategies

- *Local strategies*: statistically independent randomizations over choice sets,

$$y_{iI} \in Y_{iI} = \Delta(C_I)$$

Definition 4.5 *A* **behavior strategy** y_i *for player* i *is a function that assigns a local strategy to each information set* $I \in \mathcal{I}_i$ *of the player*

- As if players randomize as play proceeds

- Notation: $y_i \in Y_i = \times_{I \in \mathcal{I}_i} Y_{iI}$

- Behavior-strategy profiles: $y \in Y = \times_{i \in N} Y_i$

- Realization probabilities: $\hat{\rho}(\tau, y) =$ the product of all choice probabilities along τ

Definition 4.6 *The* **behavior-strategy payoff function** $\hat{\pi}_i$ *of player* i *is defined by*

$$\hat{\pi}_i(y) = \sum_{\tau \in T} \hat{\rho}(\tau, y) v_i(\tau)$$

4.4 Outcome and path

- Terminology for pure, mixed and behavior strategy profiles:

Definition 4.7 Outcome *of strategy profile = induced probability distribution over plays*

Definition 4.8 Path *of strategy profile = the set of plays assigned positive probabilities = the support of the outcome.*

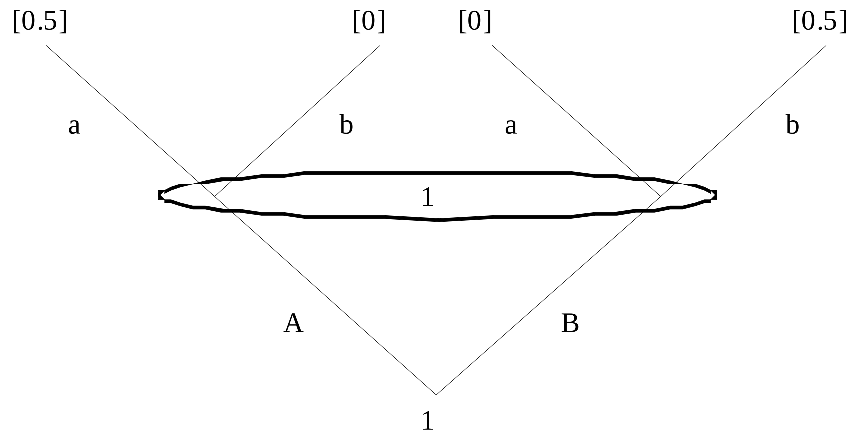
- Also applied to nodes and information sets "*on* and *off* the path".

5 Perfect recall and Kuhn's theorem

- Mixed strategies: "global randomizations" performed at the beginning of the play of the game

- Behavior strategies: "local randomizations" performed during the course of play of the game

- Equivalence in terms of realization probabilities?



Definition 5.1 (Kuhn 1950,1953) *An extensive form* Φ *has* **perfect recall** *if*

$$c < a \Leftrightarrow c < a'$$

for each player $i \in N$*, pair of information sets* $I, J \in \mathcal{I}_i$*, choice* $c \in C_I$ *and nodes* $a, a' \in J$*.*

- Note: An extensive form has perfect recall if each player has only one information set.

- Note: Bernoulli values and payoffs are irrelevant for this definition

Informally:

Theorem 5.1 ("Kuhn's Theorem") *If* Φ *has perfect recall, then, for each mixed strategy,* $\exists$ *a realization-equivalent behavior strategy.*

5.1 Behavior-strategy mixtures

To state this more exactly: Consider a player i in a finite extensive form Φ.

Definition 5.2 *A (***behavior-strategy) mixture***, w_i, is a finite-support randomization over the player's set of behavior strategies: $w_i \in W_i$, where W_i is the set of probability vectors $w_i = \left(w_i\left(y_i^1\right), ..., w_i\left(y_i^k\right)\right)$ for some $k \in \mathbb{N}$ and $y_i^1, ..., y_i^k \in Y_i$.*

- Every behavior strategy $y_i \in Y_i$ can be viewed as a (degenerate) behavior-strategy mixture, the mixture w_i that assigns unit probability to y_i.

- Every mixed strategy $x_i \in X_i$ can be viewed as the mixture w_i that assigns probability $x_{ih} \in [0, 1]$ to the (degenerate) behavior strategy y_i^h that assigns unit probability to the choices made under pure strategy $h \in S_i$.

Definition 5.3 *A mixture* $w'_i \in W_i$ *is* **realization equivalent** *with a mixture* $w_i \in W_i$ *if the realization probabilities under the profile* $\left(w'_i, w''_{-i}\right)$ *are identical with those under* $\left(w_i, w''_{-i}\right)$*, for all profiles* $w'' \in \times_{j=1}^{n} W_j$*.*

Theorem 5.2 (Kuhn 1950, Selten 1975) *Consider a player* i *in a finite extensive form* Φ *with perfect recall. For each behavior-strategy mixture* $w_i \in W_i$ *there exists a realization-equivalent mixture* $w'_i \in W_i$ *that assigns unit probability to a behavior strategy* $y_i \in Y_i$*.*

Rough proof sketch:

1. Consider those of i's information sets I that are *possible* under w_i in the sense that I is on the path of $\left(w_i, w''_{-i}\right)$ for some $w'' \in \times_{j=1}^{n} W_j$

2. Note that conditional probabilities across nodes in an information set $I \in \mathcal{I}_i$ do not depend on i's own strategy

6 Normal-form games

- A *normal-form game*: a triplet $G = (N, S, \pi)$ where

  N is the set of *players*

  $S = \times_{i \in N} S_i$ the set of *strategy profiles* $s = (s_i)_{i \in N}$, S_i the *strategy set* of player i

  $\pi : S \to \mathbb{R}^n$ is the *combined payoff function*, $\pi_i(s) \in \mathbb{R}$ the payoff to player i under s

Example 6.1 *A firm offering a wage* $w \in W = [0, 100]$ *to a worker, who can accept or reject the offer. If accept* ($y = 1$)*, then* $v_1 = 100 - w$ *(profit) and* $v_2 = w$ *(utility). If reject* ($y = 0$)*, then* $v_1 = v_2 = 0$*. The normal form:*

$S_1 = W = [0, 100]$

$S_2 = \{0, 1\}^W$ *; the set of* **functions** $f : W \to \{0, 1\}$

$\pi_1 (w, f) = (100 - w) \cdot f (w)$

$\pi_2 (w, f) = w \cdot f (w)$

7 Five NF games associated with each EF game

- G is called *finite* if both N and S are finite

- Five normal form for a given EF game Γ:

  1. The pure-strategy normal form $G = (N, S, \pi)$

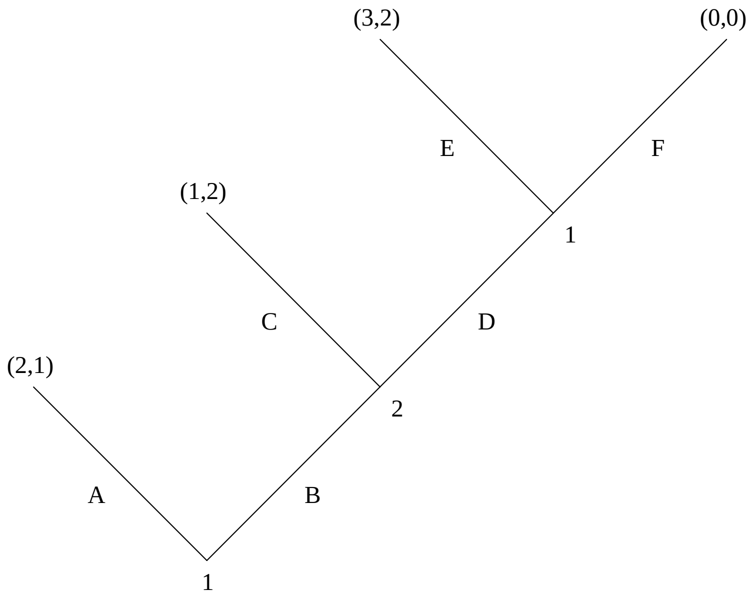
  2. The mixed-strategy normal form $\tilde{G} = (N, X, \tilde{\pi})$

  3. The behavior-strategy normal form $\hat{G} = (N, Y, \hat{\pi})$

  4. The quasi-reduced normal form

  5. The reduced normal form

Example 7.1



Game 9

	C	D
AE	2, 1	2, 1
AF	2, 1	2, 1
BE	1, 2	3, 2
BF	1, 2	0, 0

Quasi-reduced (and reduced):

	C	D
A	2, 1	2, 1
BE	1, 2	3, 2
BF	1, 2	0, 0

8 Thompson's transformations

- Thompson (1952) studied four "strategically inessential" transformations of finite extensive-form games (see also Kohlberg and Mertens, 1986).

- Thompson showed that by successive application of these transformation, any finite extensive-form game can be rendered on the form of a simultaneous-move game.

- However, one of these transformations (called inflate-deflate) may result in a game *without perfect recall*

- Elmes and Reny (1994) proved that one can dispense with that transformation if one of the other transformations is slightly modified.

- The three transformations are “add”, “coalesce” and “interchange”

1. “Add”, consists in adding a node to a player’s information set in such a way that the player’s choice will not affect any player’s payoff in case play would reach the added node. [Reconsider the entry-deterrence game in lecture 1]

2. “Coalesce” brings together two consecutive decision nodes, each being a singleton information set and belonging to the same player. [Example in class]

3. “Interchange” changes the order of moves between two players who are not informed of each others’ moves. [Reconsider Game 2 in lecture 1]

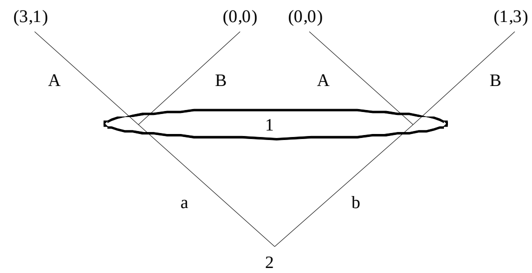


Figure 1:

Theorem 8.1 (Elmes and Reny) *If Γ and Γ' are extensive-form games with perfect recall and have the same quasi-reduced normal form, then there exists a finite sequence of games, $\Gamma_1, ..., \Gamma_k$, each with perfect recall, such that (a) $\Gamma_1 = \Gamma$ and $\Gamma_k = \Gamma'$ and (b) consecutive games in the sequence differ only by one of the transformations "add", "coalesce" or "interchange".*

- Hence, if we, as analysts, deem the three transformations "strategically inessential" then we will prefer solution concepts that are invariant under these transformations, that is, that (by the above theorem) depend only on the quasi-reduced normal form. [See discussion in Kohlberg and Mertens, 1986]

9 Solution concepts

Now we are in a position to define and analyze different solution concepts for games

- Solution concepts for extensive-form games (complicated math)
- Solution concepts for normal-form concepts (easier math)
- Interpretations of solutions: (a) rationalistic, (b) evolutionary

  Next topic:

Solution concepts for finite normal-form games.