SOLUTION CONCEPTS FOR EXTENSIVE-FORM GAMES

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- Kuhn (1950, 1953), Selten (1965, 1975), Kreps and Wilson (1982)

- Recall: pure, mixed and behavior strategies

- Focus on behavior strategies in a finite EF game $\Gamma = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p, r, u)$ with perfect recall

- Recall "Kuhn's Theorem"

1 Nash equilibrium

Let $\hat{G} = (N, Y, \hat{\pi})$ be the behavior-strategy NF representation of Γ

- Do NE in $\hat{G}$ always exist?

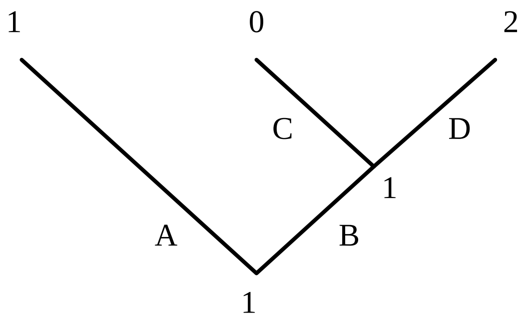
- How recognize NE in Γ?

Proposition 1.1 *There exists at least one Nash equilibrium of* $\hat{G} = (N, Y, \hat{\pi})$.

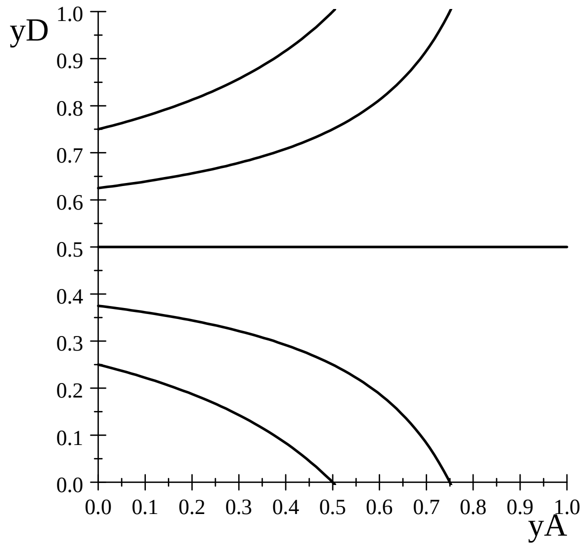
Proof: Difficulty: payoff functions in $\hat{G}$ need not be quasi-concave.

Consider instead $\tilde{G} = (N, X, \tilde{\pi})$ and use Kuhn's Theorem!

Example 1.1



A contour map for $\hat{\pi}_1(y_A, y_D)$*:*



To characterize NE directly in Γ :

1. For any node a, information set I and behavior-strategy profile y, let

$$\hat{\rho}(a, y) = \sum_{\tau \in T^a} \hat{\rho}(\tau, y)$$

$$\hat{\rho}(I, y) = \sum_{a \in I} \hat{\rho}(a, y)$$

2. Definition: $I \in \mathcal{I}$ is *on the path* of $y \in Y$ if $\hat{\rho}(I, y) > 0$

3. Consider I on the path of y. By Bayes' rule, the *conditional probabilities* of nodes $a \in I$ are

$$\eta(a \mid y) = \frac{\hat{\rho}(a, y)}{\hat{\rho}(I, y)}$$

Definition 1.1 *Suppose that* $\hat{\rho}(I, y) > 0$ *and* $I \in \mathcal{I}_i$. *A behavior strategy* $y_i^* \in Y_i$ *is a* **best reply to** y **at** I *if for all* $y_i' \in Y_i$:

$$\sum_{a \in I} \eta(a \mid y) \pi_{ia}(y_i^*, y_{-i}) \geq \sum_{a \in I} \eta(a \mid y) \pi_{ia}\left(y_i', y_{-i}\right)$$

- Here $\hat{\pi}_{ia}\left(y_i', y_{-i}\right)$ is the *conditional payoff* to player i when play starts at node a

- Note that the requirement is **not** only that the *local* strategy y_{iI}^* at I should be optimal:

Definition 1.2 *A behavior-strategy profile* $y^* \in Y$ *is* **sequentially rational on its own path** *if* $y_i^* \in Y_i$ *is a best reply to* y^* *at all* $I \in \mathcal{I}$ *with* $\hat{\rho}(I, y^*) >$ 0.

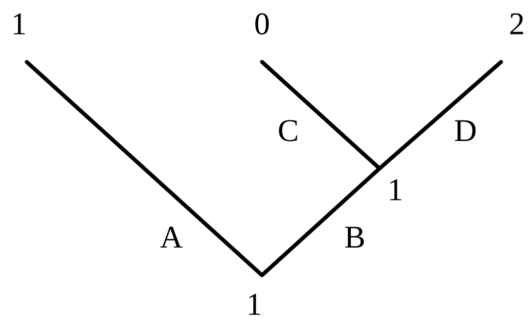


Figure 1:

Proposition 1.2 (van Damme, 1984) *A behavior-strategy profile* y^* *is a NE of* $\hat{G}$ *iff it is sequentially rational on its own path in* Γ*.*

Proof:

1. Suppose that y is *not* a NE of $\hat{G}$. Then ..

2. Suppose that y does *not* prescribe a best reply to y at some info set I on its path

- Reconsider earlier examples in the light of this characterization!

2 Subgame perfection

Definition 2.1 (Selten, 1965) *A behavior strategy profile y is a* **subgame-perfect equilibrium (SPE)** *if its restriction y^a to each subgame Γ^a is a NE of $\hat{G}^a$.*

- In simultaneous-move games: SPE=NE

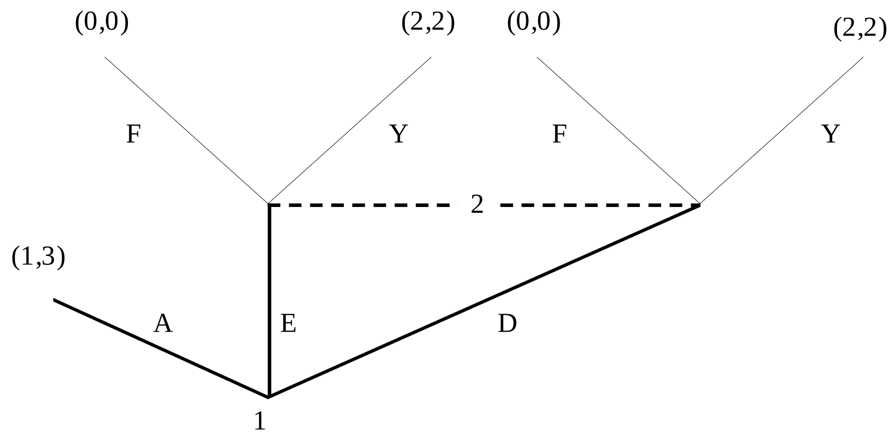
- In games of perfect information: use *Kuhn's algorithm* to construct a SPE:

Proposition 2.1 *Every finite EF game of perfect information has at least one SPE in pure strategies. For generic payoffs, this SPE is unique.*

- By way of a slight generalization of Kuhn's algorithm:

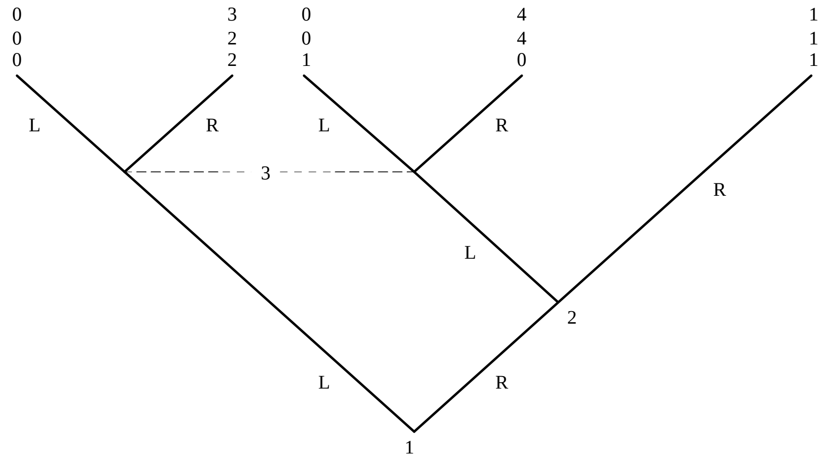
Proposition 2.2 *Every finite EF game with perfect recall has at least one SPE.*

- SPE is sensitive to details of the EF form. Reconsider the Entry–Deterrence game:



- However, this is not the only weakness of SPE:

- Subgame perfection $\not\Rightarrow$ sequential rationality at singleton-information sets ("Selten's horse"):



- $s = (L, R, R)$ is NE and hence SPE. But 2's move is not a best reply to s at 2's node

- Recall that strategies are not only contingent plans for the player in question, but also express others' expectations about the player's moves

3 Sequential equilibrium

- Refine SPE by generalizing the stochastic dynamic programming approach from 1 decision-maker to n!

- Kreps and Wilson (1982)

Definition 3.1 *A* **belief system** *is a function* $\mu : A \diagdown A_\omega \to [0, 1]$ *such that*

$$\sum_{a \in I} \mu(a) = 1 \qquad \forall I \in \mathcal{I}$$

Definition 3.2 *An belief system* μ *is* **consistent** *with a behavior-strategy profile* y *if* $\exists$ *sequence of interior behavior-strategy profiles* $y^t \to y$ *such that* $\eta\left(\cdot \mid y^t\right) \to \mu(\cdot)$.

- $\hat{\rho}(I, y) > 0 \Rightarrow \mu$ agrees with $\eta(\cdot \mid y)$ on I

Definition 3.3 *A behavior-strategy profile y is* **sequentially rational** *under a belief system μ if for every player i and each information set $I \in \mathcal{I}_i$:*

$$\sum_{a \in I} \mu(a) \pi_{ia}(y) \geq \sum_{a \in I} \mu(a) \pi_{ia}\left(y_i', y_{-i}\right) \quad \forall y_i' \in Y_i$$

Definition 3.4 *A behavior-strategy profile y is a* **sequential equilibrium (SE)** *if y is sequentially rational under some belief system μ that is consistent with y.*

Proposition 3.1 *Every sequential equilibrium is subgame perfect.*

- The SE concept was inspired by Selten's (1975) perfection criterion for EF games:

Definition 3.5 *A behavior-strategy profile y in a finite extensive-form game Γ is* **extensive-form perfect** *if it is a perfect equilibrium of the agent normal form of Γ.*

Definition 3.6 *The* **agent normal form** *associated with a finite extensive-form game Γ is the (pure strategy) normal form of the agent-decentralized extensive-form game Γ', in which each player has been split into agents, one for each of the player's information set, and assigning the player's Bernoulli function values to all his or her agents.*

- Existence of EF perfect equilibria follows immediately by Nash's existence theorem applied to the agent NF!

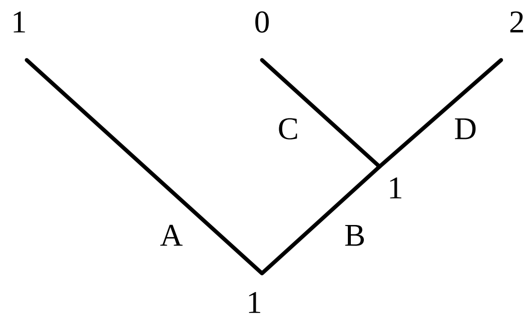


Figure 2:

- Moreover:

Proposition 3.2 *Every EF perfect equilibrium is a SE.*

Proof idea: y^* EF perfect $\Rightarrow$ y^* sequentially rational against small interior perturbations of y^*

- We obtain the existence of SE for free!

- Example of a SE that is not EF perfect?

4 A critical examination of the SE solution concept

1. The belief system is required to be the same for all players

- Interpersonal consistency may reject arguably reasonable NE:

A remedy, if one thinks that this is an unnatural restriction, is to require only that **for each player** there exists some consistent belief system (for that player)!

Definition 4.1 *A behavior-strategy profile y is a* **decentralized sequential equilibrium** *if there for each player i exists a consistent belief system μ^i under which y_i is sequentially rational for player i.*

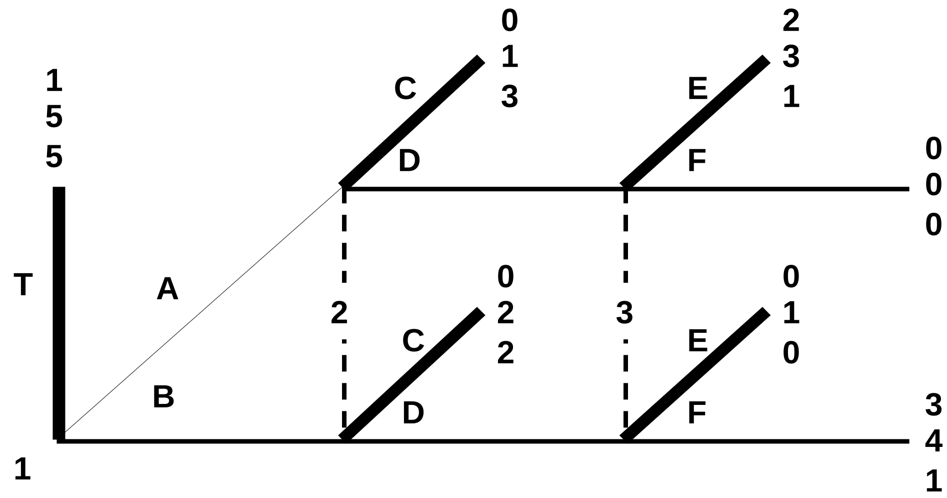


Figure 3:

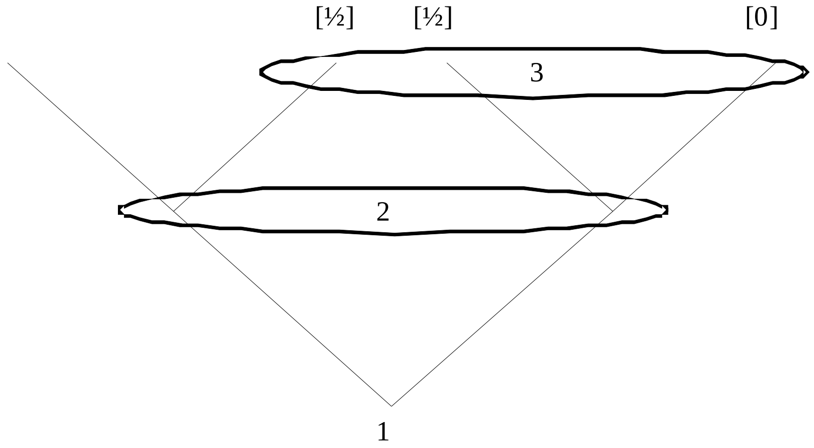
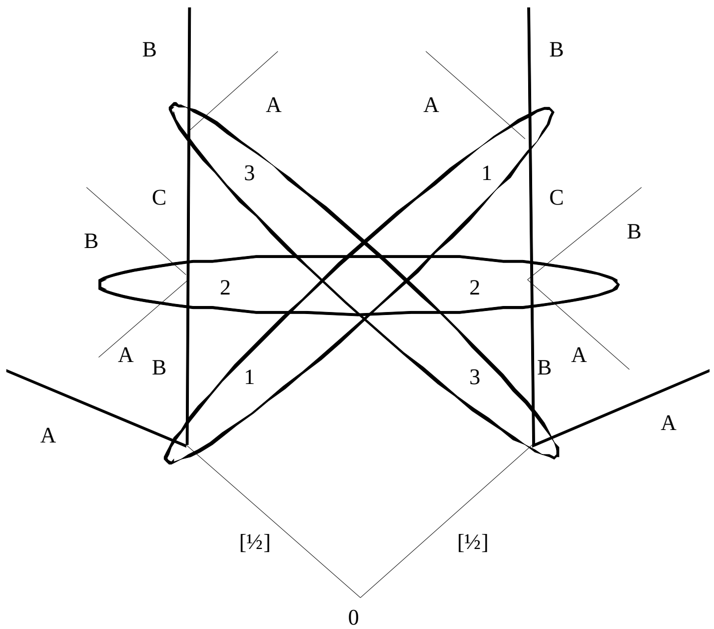


Figure 4:

2. Consistency $\not\Rightarrow$ structural consistency (Kreps and Ramey (1987)):

- But is structural consistency important? (See Kreps and Ramey, 1987)

3. **Consistency may conflict with sequential rationality** (Kreps and Ramey, 1987)



- Cognitive dissonance in the mind of player 2 if $y_1 = y_3 = A$ and $\mu(a), \mu(b) > 0$

- Remedy: restrict domain of SE to EF games with "well-ordered" information sets:

Definition 4.2 *An extensive form* $\Phi = (N, A, \psi, \mathcal{P}, \mathcal{I}, \mathcal{C}, p)$ *has* **chronologically ordered information sets** *if there exists a ("clock") function* $\tau : A \to \mathbb{R}$ *such that*

(i) $a < b \quad \Rightarrow \tau(a) < \tau(b)$ *and*

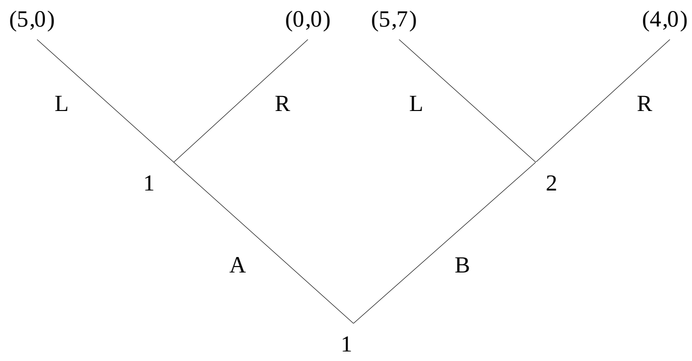
(ii) $\tau(a) = \tau(b)$ *if* a *and* b *belong to the same information set* $I \in \mathcal{I}$

5 Relations between NF and EF solutions

- Let Γ be a finite extensive-form game with perfect recall, let G be its pure-strategy normal form and $\tilde{G}$ the extensive-form extension of G.

- EF perfection $\not\Rightarrow$ NF perfection

Example 5.1



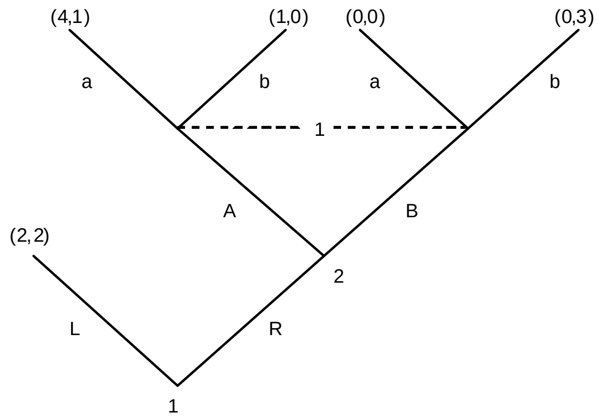
- Recall the "free insurance" argument against playing a weakly dominated strategy

  - insurance against *other* player's global trembles

- However, if you may want to also insure yourself against your *own future trembles*, then you may use certain weakly dominated strategies

- NF perfection $\not\Rightarrow$ EF perfection

Example 5.2



$s^* = (La, B)$ *an undominated NE (write up the NF!)*

(Robust against 1's tremble to Rb *and* Ra *with equal prob.)*

But s^* *is not SPE, hence not EF perfect!*

- Recall the NF-solution concept of **properness**

- Remarkable property (van Damme (1983) and Kohlberg and Mertens (1986)):

Proposition 5.1 *Let G be a finite game with mixed strategy extension $\tilde{G}$. For every proper equilibrium x^* in $\tilde{G}$ and every EF game Γ with G as its NF, there exists a realization-equivalent SE y^* in Γ.*

Proof sketch: Let G, $\tilde{G}$, Γ and x^* be as stated

1. $\exists$ a sequence of ε_t-proper profiles $x^t \in int\,[\Box\,(S)]$ with $\varepsilon_t \to 0$ and $x^t \to x^*$

2. For each x^t $\exists$ a realization-equivalent behavior-strategy y^t in Γ

3. Since $x^t \in int\,[\Box\,(S)]$, each info set in Γ is on the path of y^t

4. Since $x^t \to x^*$, $y^t \to y^* \in Y$

5. Sufficient to verify that y^* is a SE

6. For each $t \in \mathbb{N}$, let $\mu^t = \eta\left(\cdot \mid y^t\right)$

7. Then $\mu^* = \lim_{t\to\infty} \mu^t$ is a belief system consistent with y^*

8. Suppose that y^* is not sequentially rational under μ^*

9. Then $\exists$ player i, information set $I \in \mathcal{I}_i$ and $y_i' \in Y_i$ such that $y_{iI}' \neq y_{iI}^*$ and

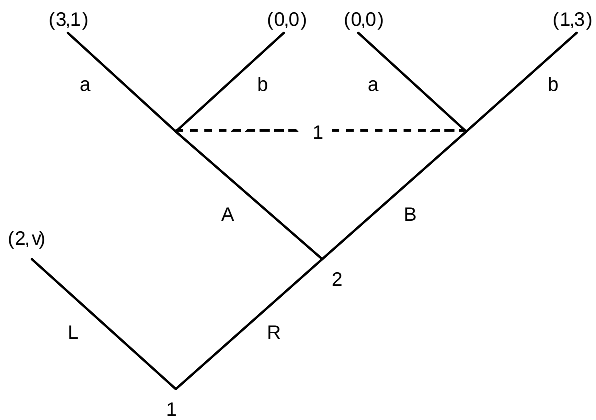
$$\sum_{a \in I} \mu^* (a) \, \hat{\pi}_{ia} \left(y_i', y_{-i}^* \right) > \sum_{a \in I} \mu^* (a) \, \hat{\pi}_{ia} \left(y^* \right)$$

10. It remains to show that this is not possible

11. This is not easy, but not impossible...

6 Interpreting deviations in games of imperfect information

- Consider a battle-of-the-sexes where player 1 has an outside option (go to a café with a friend).



- In this game, all EF solution concepts agree that $s = (Lb, B)$ is a solution

- Is s “reasonable”? How would you reason if you were player 2?

- “Forward induction” (Kohlberg-Mertens (1986), van Damme (1989))

- Only $s' = (Ra, A)$ is immune against “forward inductive” reasoning

- Logic very different in spirit from perfection: deviations interpreted as intentional rather than as unintentional (mistakes)

Next lecture: Evolutionary stability concepts and the replicator dynamic

THE END