EVOLUTIONARY STABILITY CONCEPTS

Jörgen Weibull

February 24, 2010

- Rationalistic and evolutionary paradigms

- The ”as if” approach: Alchian, Friedman

- Evolutionary theorizing: De Mandeville, Malthus, Darwin, Maynard Smith

- Darwin: exogenous environment

  - “perfect competition”

- Maynard Smith: endogenous environment

  - “imperfect competition”

- Nash’s “mass action interpretation”

- Evolutionary process $=$

  $=$ mutation process $+$ selection process

- Richard Dawkins:

  - "the selfish gene"

  - "replicator"

  - "meme"

- The unit of selection: strategies

- **Evolutionary stability**: focus on robustness to mutations

- **Replicator dynamics**: focus on selection and let dynamic stability take care of mutations

1 Definition and preliminaries

- ESS $=$ *evolutionarily stable strategy* (Maynard Smith and Price (1972), Maynard Smith (1973)

  - "robustness against behavioral mutations"

  - "a strategy that will not 'let go', once it has become the 'convention' in a population

Heuristically

1. A population of individuals who are recurrently and randomly matched in pairs to play a symmetric two-player game

2. Initially, all individuals always use the same pure or mixed strategy, x

3. Suddenly, a small population share switch to another pure or mixed strategy, y

4. If the *incumbents* (those who play x) on average do better — in terms of their average payoff — than the *mutants* (those who play y), then x is said to be *evolutionarily stable against* y

5. x is *evolutionarily stable* if it is evolutionarily stable against *all* mutations $y \neq x$

Domain of analysis

- *Symmetric finite two-player games in normal form*

Definition 1.1 *A game* $G = (N, S, \pi)$ *is a* **finite and symmetric two-player game** *if* $N = \{1, 2\}$, $S_1 = S_2 = S = \{1, ..., m\}$ *and* $\pi_2(h, k) = \pi_1(k, h)$ *for all* $h, k \in S$

- Payoff bimatrix (a_{hk}, b_{hk})

- Symmetry $\Leftrightarrow$ $B = A^T$

- Write $\mathbf{\Delta}$ for $\mathbf{\Delta}(S)$, the set of mixed strategies:

$$\mathbf{\Delta} = \{x \in \mathbb{R}_+^m : \sum_{i \in S} x_i = 1\}$$

- Write the payoff to any strategy $x \in \Delta$ when used against any strategy $y \in \Delta$ (irrespective of player roles):

$$u(x, y) = x \cdot Ay$$

- While Prisoner Dilemma games are symmetric, the Matching-Pennies game is not

Example 1.1 (PD) *Here* $B = A^T$:

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 2,2 \end{array} \qquad A = \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$$

Example 1.2 (MP) *Here* $B \neq A^T$:

$$\begin{array}{ccc} & H & T \\ H & 1,-1 & -1,1 \\ T & -1,1 & 1,-1 \end{array} \qquad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Example 1.3 (CO) *Here* $B = A^T = A$:

$$\begin{array}{ccc} & L & R \\ L & 2,2 & 0,0 \\ R & 0,0 & 1,1 \end{array} \qquad A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

a **doubly** *symmetric game* ($A^T = A$)

- Best replies to $x \in \mathbf{\Delta}$:

$$\beta^*(x) = \{x^* \in \mathbf{\Delta} : u(x^*, x) \geq u\left(x', x\right) \ \forall x' \in \mathbf{\Delta}\}$$

- This defines a correspondence from $\mathbf{\Delta}$ to itself: $\beta^* : \mathbf{\Delta} \rightrightarrows \mathbf{\Delta}$

- Let

$$\mathbf{\Delta}^{NE} = \{x \in \mathbf{\Delta} : x \in \beta^*(x)\}$$

- Note $x \in \mathbf{\Delta}^{NE} \Leftrightarrow (x, x) \in \square^{NE}$ a symmetric NE

Proposition 1.1 *$\mathbf{\Delta}^{NE} \neq \varnothing$.*

Proof: Apply Kakutani's Fixed-Point Theorem to the correspondence β^*.

- Note: The set Δ^{NE} is compact and consists of finitely many connected and closed subsets ($\Delta^{NE} = \square^{NE} \cap D$)

- We are now in a position to define evolutionary stability:

Definition 1.2 *$x \in \Delta$ is an* **evolutionarily stable strategy (ESS)** *if for every strategy $y \neq x$ $\exists$ $\bar{\varepsilon} \in (0,1)$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$:*

$$u\left[x, \varepsilon y + (1-\varepsilon)x\right] > u\left[y, \varepsilon y + (1-\varepsilon)x\right]. \tag{1}$$

- Population mixture:

$$p = \varepsilon y + (1-\varepsilon)x \in \Delta$$

- Let $\Delta^{ESS} \subset \Delta$ denote the set of ESSs

- Note that an ESS has to be a *best* reply to itself: if $x \in \Delta^{ESS}$ then

$$u(y,x) \leq u(x,x) \ \forall y \in \Delta$$

- Hence $\Delta^{ESS} \subset \Delta^{NE}$

- Note also that an ESS has to be a *better* reply to its alternative best replies than they are to themselves: if $x \in \Delta^{ESS}$, $y \in \beta^* (x)$ and $y \neq x$, then $u(x,y) > u(y,y)$

Proposition 1.2 $x \in \Delta^{ESS}$ *if and only if for all* $y \neq x$*:*

$$u(y, x) \leq u(x, x) \tag{2}$$

$$u(y, x) = u(x, x) \Rightarrow u(y, y) < u(x, y) \tag{3}$$

- Note that some games have no ESS. Example?

Example 1.4 (PD)

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 2,2 \end{array}$$

$$\Delta^{ESS} = \Delta^{NE} = \{D\}$$

Example 1.5 (CO)

$$\begin{array}{ccc} & A & B \\ A & 2,2 & 0,0 \\ B & 0,0 & 1,1 \end{array}$$

$$\Delta^{NE} = \left\{A, B, \frac{1}{3}A + \frac{2}{3}B\right\}$$

$$\Delta^{ESS} = \{A, B\}$$

We finally got rid of the mixed NE!

Example 1.6 (3 by 3) *Recall the initial example with a unique and strict NE, where all strategies were seen to be rationalizable:*

	L	C	R
T	$7,0$	$2,5$	$0,7$
M	$5,2$	$3,3$	$5,2$
B	$0,7$	$2,5$	$7,0$

This game is not symmetric. However, we obtain a symmetric game by letting "nature" first choose who will be the row player and the column player, respectively, with equal probability for both alternatives. In this new, larger and symmetric game, each player has nine pure strategies, $S = \{TL, TC, ..., BC, BR\}$*, all rationalizable. The unique NE is for both to play* MC*. Since* MC *is the unique best reply to itself, this is the unique ESS. Hence, in this example, evolutionary stability picks a unique strategy while rationalizability permits all.*

Example 1.7 (RSP) *The rock-scissors-paper game:* $B = A^T$ *and*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

2 Maynard Smith's and Price's original example

1. Each player has two pure strategies; *"hawk"*, H, or *"dove"*, D (originally, "mouse")

2. H obtains payoff $v > 0$ when played against D, and D then obtains 0.

3. If both play H: each player has an equal chance of winning v, and the cost of losing a fight is $c > v$

   - hence, the expected payoff to strategy H against itself is $(v - c)/2$

4. If both play D: they split even and each gets payoff $v/2$.

5. The resulting payoff matrices:

$$A = \begin{pmatrix} v/2 - c/2 & v \\ 0 & v/2 \end{pmatrix} \text{ and } B = A^T$$

6. The game has two *asymmetric* pure NE: (H, D) and (D, H)

7. It also has one symmetric equilibrium: both play H with probability v/c

8. Let

$$x^* = \left(\frac{v}{c}, 1 - \frac{v}{c}\right)$$

and note that $\Delta^{NE} = \{x^*\}$

9. Since $\Delta^{ESS} \subset \Delta^{NE}$ it only remains to see if x^* is an ESS

10. Since $x^* \in \Delta^{NE} \cap int\,(\Delta)$, all $y \in \Delta$ are alternative best replies

11. Hence sufficient to show

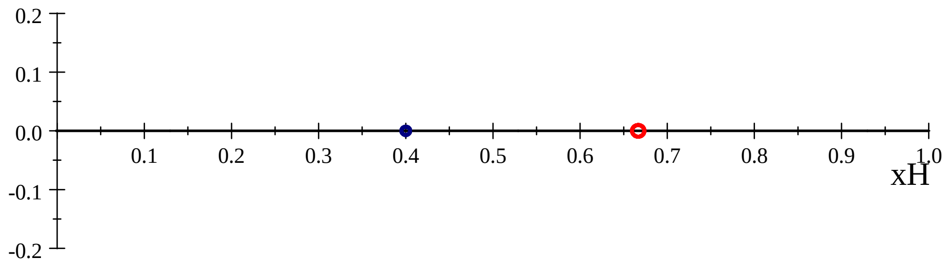
$$u(y, y) < u(x^*, y)\ \forall y\ \neq x^*$$

Two polynomials in one variable y...

12. Example: $v = 4$ and $c = 6$:

$$\begin{array}{ccc} & H & D \\ H & (-1,-1) & (4,0) \\ D & (0,4) & (2,2) \end{array}$$

$$\Delta^{NE} = \left\{ \frac{2}{3} H + \frac{1}{3} D \right\}$$



3 Some properties of ESS

3.1 Finiteness

Proposition 3.1 (Haigh, 1975) *The set Δ^{ESS} is finite.*

Proof idea: Non-overlapping supports.

3.2 Relations to non-cooperative solutions

- $x \in \Delta^{ESS} \Rightarrow x$ undominated

- $x \in \Delta^{ESS} \Rightarrow (x, x) \in \Box^{PE}$

- $x \in \Delta^{ESS} \Rightarrow (x, x) \in \Box^{PR}$ (proper)

3.3 Uniform invasion barriers

- The definition of ESS requires that, for each potential mutant strategy $y \neq x$, $\exists$ “invasion barrier” $\bar{\varepsilon}_y > 0$

- $\exists$ a *uniform* invasion barrier $\bar{\varepsilon} > 0$?

- Important question, since real populations are finite, and thus we need $\bar{\varepsilon}_y \geq 1/N \ \forall y \neq x$

Definition 3.1 $x \in \Delta$ **has a uniform invasion barrier** *if there is some* $\bar{\varepsilon} > 0$ *such that inequality (1) holds for all strategies* $y \neq x$ *and all* $\varepsilon \in (0, \bar{\varepsilon})$*.*

Proposition 3.2 $x \in \Delta^{ESS}$ *iff* x *has a uniform invasion barrier.*

3.4 Local superiority

- An *interior* ESS x earns a higher payoff against all $y \neq x$ than these earn against *themselves*

- A form of "global superiority"

- What about "local superiority"?

Definition 3.2 $x \in \Delta$ *is* **locally superior** *if it has a nbd* A *s.t.* $u(x, y) > u(y, y)$ $\forall y \neq x$, $y \in A$.

Proposition 3.3 $x \in \Delta^{ESS} \Leftrightarrow x$ *is locally superior.*

4 Neutral stability and evolutionarily stable sets

Definition 4.1 $x \in \Delta$ *is a* **neutrally stable strategy (NSS)** *if for every strategy* $y \in \Delta$ $\exists$ $\bar{\varepsilon} \in (0,1)$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon})$*:*

$$u\left[x, \varepsilon y + (1-\varepsilon)x\right] \geq u\left[y, \varepsilon y + (1-\varepsilon)x\right].$$

- Reconsider the rock-scissors-paper game

- $\Delta^{ESS} \subset \Delta^{NSS} \subset \Delta^{NE}$

- There are games with no NSS

- Set-wise evolutionary stability:

Definition 4.2 (Thomas, 1985) *A non-empty and closed set $X \subset \Delta^{NE}$ is an* **evolutionarily stable set** *(an* **ES set***) if there for each $x \in X$ exists some $\delta > 0$ such $u(x, y) \geq u(y, y)$ for all $y \in \beta^*(x)$ within distance δ from x, with strict inequality if $y \notin X$.*

- All strategies x in an ES set X are NSS

- If $x \in \Delta^{ESS}$, then $X = \{x\}$ is an ES set

- More generally:

  (i) $X \subset \Delta^{ESS} \Rightarrow X$ is an ES set

  (ii) X, X' ES sets $\Rightarrow X \cup X'$is an ES set

- There are games with no ES set (for example rock-scissors-paper)

- There are games with interesting ES sets (for example cheap-talk games)

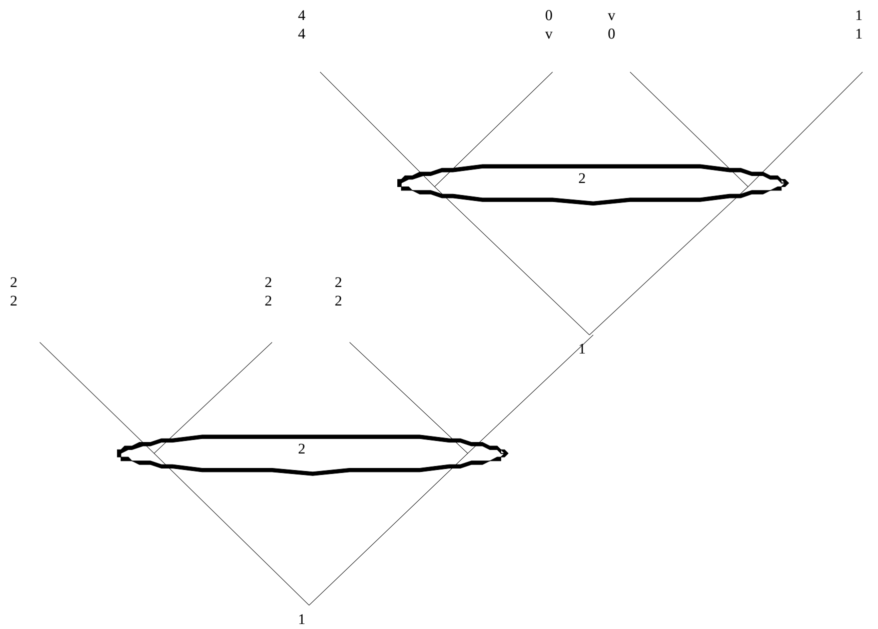
5 Equilibrium-evolutionary stable sets

Definition 5.1 (Swinkels, 1993) *A set $X \subset \square^{NE}$ is* **equilibrium evolutionarily stable (EES)** *if it is minimal with respect to the following property: X is a non-empty and closed subset of Δ^{NE} for which $\exists\, \bar{\varepsilon} \in (0,1)$ such that if $x \in X$, $y \in \Delta$, $\varepsilon \in (0, \bar{\varepsilon})$ and $y \in \tilde{\beta}\,[\varepsilon y + (1-\varepsilon)x]$, then $\varepsilon y+(1-\varepsilon)x \in X$.*

Proposition 5.1 (Swinkels, 1993) *Every EES set $X \subset \Delta^{NE}$ is a component of Δ^{NE}.*

Proposition 5.2 *Every ES set contains some EES set. Any connected ES set is an EES set.*

Example 5.1 *Entry to a prisoner's dilemma, with veto right (set $v = 5$):*



Next lecture

-the replicator and other selection dynamics

-in symmetric two-player games and in arbitrary finite games

- chapter 8 in lecture notes, chapters 3 and 5 in book

THE END