THE REPLICATOR DYNAMIC

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Two remaining questions from last lecture:

1. Symmetric version of asymmetric 3×3 example?

2. An ESS that is non-robust against multiple mutations?

Recall that:

- Evolutionary process = mutation process + selection process

- *Evolutionary stability*: focus on robustness to mutations, selection dynamics implicit

- *Replicator dynamics:* focus on selection, robustness to mutations by way of dynamic stability

[Today's material is covered in Chapters 3 and 5 in Weibull (1995). See also lecture notes.]

1 The replicator dynamic

- Domain of analysis the same as for ESS: finite and symmetric two-player games

Heuristically:

1. A population of individuals who are recurrently and randomly matched in pairs to play the game

2. Individuals use only *pure strategies* (like in Nash's mass-action interpretation)

3. A mixed strategy is now interpreted as a *population state*, a vector of populations shares

4. Population shares change, depending on the *current average payoff* to each pure strategy

5. The changes are described by a *system of ordinary differential equations*

Formally:

A game $G = (N, S, \pi)$ with $N = \{1, 2\}$, $S_1 = S_2 = S = \{1, ..., m\}$ and $\pi_2(h, k) = \pi_1(k, h)$ for all $h, k \in S$

- Payoff bimatrix (A, B) with elements (a_{hk}, b_{hk})

- Symmetry: $B = A^T$

- The state space:

$$\Delta = \{x \in \mathbb{R}^m_+ : \sum_{h \in S} x_h = 1\}$$

- The *average payoff* to any pure strategy h in any population state $x \in \Delta$:

$$u(e^h, x) = e^h \cdot Ax$$

- The *replicator dynamic* (Taylor and Jonker, 1978):

$$\dot{x}_h(t) = \left(u\left[e^h, x(t)\right] - u\left[x(t), x(t)\right]\right) \cdot x_h(t) \qquad \forall h \in S, t \in \mathbb{R}$$

1.1 Deriving the replicator dynamic

- In a finite population, let $N_h(t) \geq 0$ be the number of individuals who currently use pure strategy $h \in S$

- Let $N(t) = \sum_{h \in S} N_h(t) > 0$ be the total population

- *Population state*: $x(t) = (x_1(t), ..., x_m(t))$, where $x_h(t) = N_h(t)/N(t)$

- Thus $x(t) \in \Delta$, a mixed strategy

- *Birth-death process:*

$$\dot{N}_h = \left[\beta + u(e^h, x) - \delta\right] N_h \qquad \forall h \in S$$

$\Rightarrow$ the replicator dynamic

Proof: take time derivative of the identity

$$N(t)x_h(t) = N_h(t) \qquad \forall t \in \mathbb{R}$$

- *Growth rate* of (positive) population share of "h-strategists" $=$ the *excess payoff* to pure strategy h:

$$\frac{\dot{x}_h}{x_h} = u(e^h, x) - u(x, x)$$

- Better-than-average strategies grow

- *Best* replies have the highest growth rate

1.2 Invariance under payoff transformations

1. Replicator dynamic *orbits* are invariant under positive affine transformations of payoffs (the speed along the orbits depend on the scaling parameter)

2. Replicator dynamics *orbits* and *trajectories* are invariant under local payoff shifts (addition of subtraction of any constant to a column of A)

2 Systems of ODEs - a reminder

1. System of autonomous, first-order ordinary differential equations (ODEs):

$$\dot{x}(t) = f[x(t)]$$

where $X \subset \mathbb{R}^m$, $f : X \to \mathbb{R}^m$ and

$$\dot{x} = (\dot{x}_1, .., \dot{x}_m) = \frac{dx}{dt} = (\frac{dx_1}{dt}, .., \frac{dx_m}{dt})$$

2. x is called a *state*, X the *state space* and f the *vector field*

3. A (local) *solution* through a point $x^o \in X$ to (**??**) is a function $\xi(\cdot, x^o) : T \to X$, where T is an open interval containing $t = 0$, such that $\xi(0, x^o) = x^o$, and

$$\frac{d}{dt}\xi(t, x^o) = f[\xi(t, x^o)] \qquad \forall t \in T$$

The solution is called *global* if $T = \mathbb{R}$

4. *The Picard-Lindelöf theorem*: If f is Lipschitz continuous, then (**??**) has a unique local solution $\xi(\cdot, x^o) : T \to X$ through each point $x^o \in X$.

5. Extension of time domain: Suppose $C \subset X$ is compact and such that

$$x^o \in C \;\Rightarrow\; \exists T(x^o) \text{ open s.t. } \xi(t, x^o) \in C \; \forall t \in T(x^o)$$

Then one can prove that $\exists$ unique global solution $\xi(\cdot, x^o) : \mathbb{R} \to C$ through each $x^o \in C$ [Hale, 1969]

6. The induced global mapping $\xi : \mathbb{R} \times C \to C$ is continuous and satisfies

$$\begin{cases} \xi(0, x) = x \quad \forall x \in C \\ \xi[t, \xi(s, x)] = \xi(t + s, x) \quad \forall x \in C, \; \forall s, t \in \mathbb{R} \end{cases}$$

7. The *trajectory* $\tau(x^o)$ through $x^o \in C$ is the *graph* of the solution through x^o:

$$\tau(x^o) = \{(t, x) \in \mathbb{R} \times C : x = \xi(t, x^o)\}$$

8. The *orbit* $\gamma(x^o)$ through x^o is the *range* of the solution through x^o:

$$\gamma(x^o) = \{x \in C : x = \xi(t, x^o) \text{ for some } t \in \mathbb{R}\}$$

9. A subset $A \subset C$ is *invariant* if $\gamma(x^o) \subset A$ for all $x^o \in A$.

10. If $A \subset C$ is invariant, then so is $\bar{A} \subset C$, $B = C \cap \sim A$, $int(A) \subset C$ and $bd(A) \subset C$

11. A *stationary (or equilibrium) state* under ξ is a state $x \in C$ such that $\xi(t, x) = x$ for all $t \in \mathbb{R}$

12. The Picard-Lindelöf Theorem $\Rightarrow$ unique solution through every stationary state, *so if you are not in equilibrium, you will never be*....

13. Proposition: If $\lim_{t\to+\infty} \xi(t,x) = x^*$, then x^* is stationary

14. The *forward orbit* $\gamma^+(x^o)$ through x^o:

$$\gamma^+(x^o) = \{x \in C : x = \xi(t,x^o) \text{ for some } t \geq 0\}$$

15. A subset $A \subset C$ is *forward invariant* if $\gamma^+(x^o) \subset A$ for all $x^o \in A$

16. A state $x \in C$ is (*Lyapunov) stable* if every nbd B of x contains a nbd B^o of x s.t.

$$x^o \in B^o \cap C \;\Rightarrow\; \xi(t,x^o) \in B \;\forall t \geq 0$$

17. A state $x \in C$ is *(locally) asymptotically stable* if it is stable and $\exists$ a nbd A of x s.t.

$$\lim_{t \to +\infty} \xi(t, x^o) = x \quad \forall x^o \in A \cap C$$

3 Results for the replicator dynamic

Proposition 3.1 *Strictly dominated pure strategies are asymptotically wiped out from the population: If $k \in S$ is strictly dominated by some strategy $y \in \Delta$, then*

$$\lim_{t \to +\infty} \xi_h(t, x^o) = 0 \quad \forall x^o \in int\,(\Delta)$$

Proof:

1. Suppose $k \in S$ is strictly dominated by $y \in \Delta$

2. Then

$$\min_{x \in \Delta} \left[u\,(y, x) - u(e^k, x) \right] = \varepsilon > 0$$

3. Let $V : int(\Delta) \to \mathbb{R}$ be defined by

$$V(x) = \sum_{h \in S} y_h \ln(x_h) - \ln(x_k)$$

4. Then V increases along the replicator solution trajectories:

$$\dot{V}(x) = \sum_{h \in S} \frac{\partial V(x)}{\partial x_h} \dot{x}_h = \sum_{h \in S} \frac{y_h \dot{x}_h}{x_h} - \frac{\dot{x}_k}{x_k}$$

$$\begin{aligned} &= \sum_{h \in S} y_h \cdot \left[u(e^h, x) - u(x, x) \right] - \left[u(e^k, x) - u(x, x) \right] \\ &= u(y, x) - u(e^k, x) \geq \varepsilon \quad \forall x \in \Delta \end{aligned}$$

5. Hence, $V(x) \to +\infty$ and thus $x_k \to 0$

- The result can easily be generalized to any pure strategy that is *iteratively* strictly dominated

- Since those strategies are not rationalizable:

Proposition 3.2 *If* $h \in S$ *is not rationalizable, then it is asymptotically wiped out, if initially all pure strategies are present in the population:*

$$\lim_{t \to +\infty} \xi_h(t, x^o) = 0 \quad \forall x^o \in int\,(\Delta)$$

- In the limit, it is *as if* CK[game+rationality] would hold in the population

- Note that the two propositions are true irrespective of whether the solution trajectory converge or not

- If a solution trajectory does converge, and all pure strategies are initially present, we obtain more:

Proposition 3.3 *If* $x^o \in int\,(\Delta)$ *and* $\lim_{t\to+\infty} \xi\,(t, x^o) = x$*:* $x \in \Delta^{NE}$*.*

- We obtained NE without any rationality or knowledge assumption!

Proof of proposition:

1. Suppose $x^o \in int(\Delta)$ and $\xi(t, x^o)_{t\to+\infty} \to x$ but $x \notin \Delta^{NE}$

2. $\exists\ h \in S$ such that $u(e^h, x) - u\,(x, x) = \varepsilon > 0$

3. Since $\xi(t, x^o) \to x$ and u is continuous: $\exists T > 0$ such that

$$u\left[e^h, \xi\left(t, x^o\right)\right] - u\left[\xi(t, x^o), \xi(t, x^o)\right] > \frac{\varepsilon}{2} \quad \forall t \geq T$$

4. By the replicator dynamic:

$$\dot{x}_h = \left[u\left(e^h, x\right) - u\left(x, x\right)\right] x_h > \frac{\varepsilon}{2} \cdot x_h \quad \forall t \geq T$$

so $\xi_h(t, x^o) \to +\infty$, a contradiction to $\xi\left(t, x^o\right) \to x$

- Note that every $x \in \Delta^{NE}$ is *stationary* in the replicator dynamic

- However, not every $x \in \Delta^{NE}$ is *dynamically stable*. But:

Proposition 3.4 *If $x \in \Delta$ is Lyapunov stable, then $x \in \Delta^{NE}$.*

Proof sketch:

1. Suppose that x^* is stationary in the replicator dynamic, but $x^* \notin \Delta^{NE}$

2. Then all pure strategies in the support of x^* earn the same suboptimal payoff against x^*

3. Thus $\exists\ h \in S$ such that $x_h^* = 0$ and $u(e^h, x^*) - u(x^*, x^*) = \varepsilon > 0$

4. $\exists\ \delta > 0$ s.t. $\|x - x^*\| < \delta \Rightarrow u(e^h, x) - u(x, x) > \varepsilon/2$

5. This defines a nbd B of x^*, and $\|x - x^*\| < \delta/2$ defines a sub-nbd $B_o \subset B$

6. For all $x \in B_o$: $\dot{x}_h > (\varepsilon/2)\, x_h$, contradicting $\xi_h(t, x^o) \to 0$

Proposition 3.5 *If x is asymptotically stable, then $x \in \Delta^{NE}$ is undominated.*

- We obtained PE without any bounded rationality assumption!

Proposition 3.6 *If $x \in \Delta^{ESS}$, then x is asymptotically stable. The converse holds for 2×2 games, but not generally for larger games.*

Proof sketch:

1. Suppose $x \in \Delta^{ESS}$

2. Let $S(x) \subset S$ be its support and let $\Delta(x) = \{y \in \Delta : y_h > 0 \ \forall h \in S(x)\}$

3. Define $V : \Delta(x) \to \mathbb{R}$ by

$$V(y) = \sum_{h \in S(x)} x_h \ln y_h$$

and note that

$$\arg\max_{y \in \Delta(x)} V(y) = \{x\}$$

4. Along replicator solution trajectories:

$$\begin{aligned} \dot{V}(y) &= \sum_{h \in S} x_h \dot{y}_h / y_h = \sum_{h \in S} x_h \cdot \left[u(e^h, y) - u(y, y) \right] \\ &= u(x, y) - u(y, y) \end{aligned}$$

5. $x \in \Delta^{ESS} \Rightarrow x$ locally superior on a nbd B of x, so $\dot{V}(y) > 0 \ \forall y \in B$

6. Also $B \cap \Delta(x)$ is a nbd of x, and $lim_{t \to +\infty} \xi(t, y) = x \ \forall y \in B \cap \Delta(x)$

- Counter-example in class: An asymptotically stable $x \notin \Delta^{ESS}$

Proposition 3.7 *If $x \in \Delta^{NSS}$, then x is Lyapunov stable.*

Proof (surprisingly) hard! [Bomze and Weibull, 1995]

4 Examples

Example 4.1 (PD) *Prisoner's dilemma game*

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 2,2 \end{array}$$

$$\mathbf{\Delta}^{ESS} = \mathbf{\Delta}^{NE} = \{D\}$$

Example 4.2 (CO) *Coordination game*

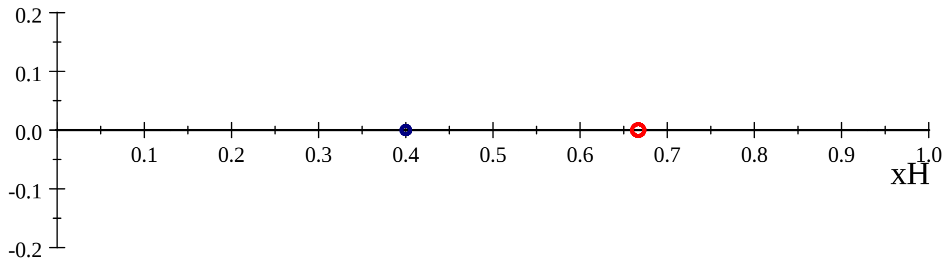
$$\begin{array}{ccc} & A & B \\ A & 2,2 & 0,0 \\ B & 0,0 & 1,1 \end{array}$$

$$\Delta^{NE} = \left\{ A, B, \frac{1}{3}A + \frac{2}{3}B \right\}, \; \Delta^{ESS} = \{A, B\}$$

Note history dependence!

Example 4.3 *Hawk-dove game*

$$A = \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$



Example 4.4 (RSP) *The rock-scissors-paper game*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

All solution trajectories, starting from any interior $x^o \neq x^*$*, are periodic, circling around* x^**. Verify that the solution orbits are of the form* $x_1 x_2 x_3 = c$*, for constants* $c \in [0, 1]$*.*

5 Multi-population dynamics

- Domain: arbitrary finite games in normal form, $G = (N, S, \pi)$, with mixed-strategy extensions, $\tilde{G} = (N, \square(S), \tilde{\pi})$

5.1 Generalizing the replicator dynamic

- A population for each player role $i \in N$

- A mixed-strategy profile $x = (x_1, ..., x_n) \in \square$ viewed as a *population state*

- The (Taylor, 1979) *multi-population replicator dynamic*:

$$\dot{x}_{ih} = \left[\tilde{\pi}_i(e_i^h, x_{-i}) - \tilde{\pi}_i(x) \right] x_{ih} \qquad \forall i \in N, h \in S_i, x \in \square$$

  (time argument t suppressed)

- The vector field is still Lipschitz continuous

- The state space is still compact

- $\exists!$ global solution $\xi(\cdot, x^o) : \mathbb{R} \to \square$ through any $x^o \in \square$

  $\xi(t, x^o)$ is the state at time $t \in \mathbb{R}$, given the initial state $x^o \in \square$ at $t = 0$

- Each of $\square$, $int\,(\square)$ and $\partial\square$ are invariant in this dynamic

- Set of *stationary states*:

$$\square^o = \left\{ x \in \square : \tilde{\pi}_i(e_i^h, x_{-i}) = \tilde{\pi}_i(x) \ \ \forall i \in N,\ h \in \mathsf{supp}\,(x_i) \right\}$$

- Thus:

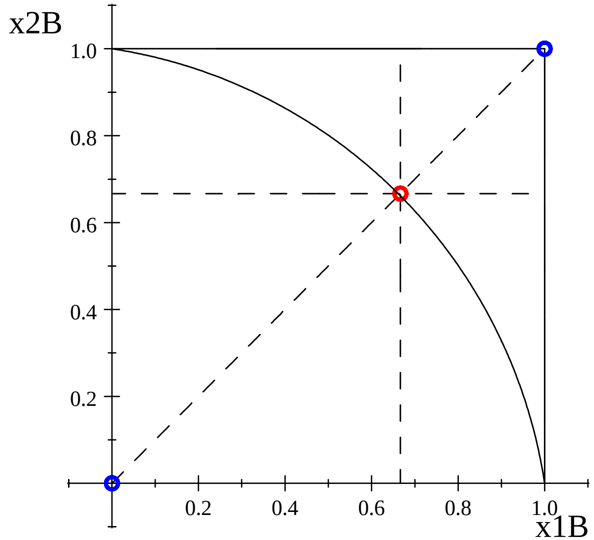
$$\square^o \cap int(\square) \subset \square^{NE} \subset \square^o$$

5.2 Examples

Example 5.1 (Prisoner's dilemma game) *Clearly, as $t \to +\infty$:*

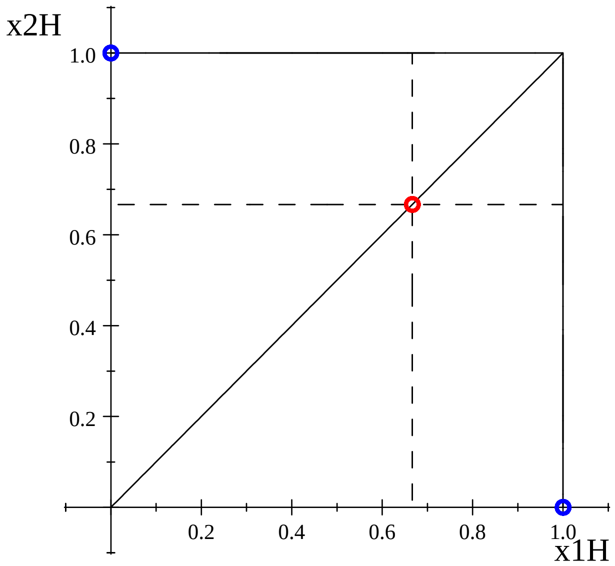
$$\xi_{iC}(t, x^o) \to 0 \qquad \textit{for } i = 1, 2 \textit{ and any } x^o \in int(\Box)$$

Example 5.2 (Coordination game)



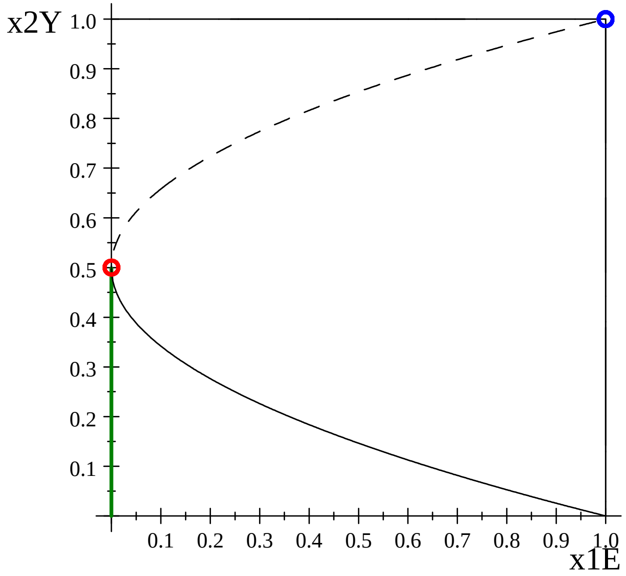
Note (a) **instability** *of the mixed NE and (b) the* **history dependence**

Example 5.3 (Hawk-dove game)



Note (a) **instability** *of the mixed NE (!) and (b) the different* **history dependence**

Example 5.4 (Entry deterrence game)



The population state may converge to "no entry" and each such state, with $x_{2Y} < 1/2$ is Lyapunov stable

But perpetual small population shocks (random mutations) will eventually take it to (E, Y)

Next lecture March 22: First about general deterministic selection dynamics and then about stochastic population dynamics

- Lecture notes
- Benaim and Weibull (2003)
- Young (1993, 1998)

THE END