DETERMINISTIC AND STOCHASTIC SELECTION DYNAMICS

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1 The multi-population replicator dynamic

- Domain of analysis: finite games in normal form, $G = (N, S, \pi)$, with mixed-strategy extensions, $\tilde{G} = (N, \square(S), \tilde{\pi})$

- For each player role $i \in N$: a (continuum) population of individuals

- All individuals use pure strategies, but may shift from one pure strategy to another, depending how well they do

- A mixed-strategy profile $x = (x_1, ..., x_n) \in \square(S)$ interpreted as a *population state*

- With the time argument t suppressed:

$$\dot{x}_{ih} = \left[\tilde{\pi}_i(e_i^h, x_{-i}) - \tilde{\pi}_i(x)\right] \cdot x_{ih} \qquad \forall i \in N, h \in S_i, x \in \square$$

1.1 Results

Proposition 1.1 (Samuelson and Zhang, 1992) *If a pure strategy* $h \in S_i$ *is iteratively strictly dominated (by a pure or mixed strategy), then*

$$x^0 \in int(\square) \;\Rightarrow\; \lim_{t \to +\infty} \xi_{ih}\left(t, x^0\right) = 0$$

Proposition 1.2 (Nachbar, 1990) *If an interior solution trajectory converges, then its limit point is a Nash equilibrium:*

$$\left[x^0 \in int(\square) \;\wedge\; \lim_{t \to +\infty} \xi\left(t, x^0\right) = x\right] \;\Rightarrow\; x \in \square^{NE}$$

Proposition 1.3 (Bomze, 1986) *If* $x \in \square(S)$ *is Lyapunov stable, then* $x \in \square^{NE}$*.*

Proposition 1.4 (Ritzberger and Weibull, 1995) $x \in \square(S)$ *is asymptotically stable iff* x *is a strict NE.*

2 General selection dynamics

Consider n-population dynamics of the form

$$\dot{x}_{ih} = g_{ih}(x)x_{ih} \qquad \forall i \in N, h \in S_i, x \in \square$$

where g is regular:

Definition 2.1 $g : \square(S) \to \mathbb{R}^m$ *is a* **regular growth-rate function** *if it is locally Lipschitz continuous and* $g_i(x) \cdot x_i = 0$ *for all* $i \in N, x \in \square(S)$*.*

Here $m = |S_1| + ... + |S_n|$.

- This guarantees the existence and uniqueness of solutions, by the Picard-Lindelöf Theorem, and that $\square(S)$ is invariant

2.1 Classes of growth-rate functions

- *Payoff monotonicity (PM):*

$$\tilde{\pi}_i(e_i^h, x_{-i}) > \tilde{\pi}_i(e_i^k, x_{-i}) \;\Leftrightarrow\; g_{ih}(x) > g_{ik}(x)$$

- *Payoff positivity (PP):*

$$sign\,[g_{ih}(x)] = sign\left[\tilde{\pi}_i(e_i^h, x_{-i}) - \tilde{\pi}_i(x)\right]$$

- *Weak payoff-positivity (WPP):*

let $B_i\,(x) = \left\{h \in S_i : \tilde{\pi}_i(e_i^h, x_{-i}) > \tilde{\pi}_i(x)\right\}$ and require

$$B_i\,(x) \neq \varnothing \;\Rightarrow\; g_{ih}(x) > 0 \text{ for some } h \in B_i\,(x)$$

- *Convex payoff monotonicty (CPM):*

$$\tilde{\pi}_i(y_i, x_{-i}) > \tilde{\pi}_i(e_i^k, x_{-i}) \;\Leftrightarrow\; y_i \cdot g_i(x) > g_{ik}(x)$$

2.2 Examples

1. The (Taylor, 1979) multi-population replicator dynamic:

$$g_{ih}(x) = \tilde{\pi}_i(e_i^h, x_{-i}) - \tilde{\pi}_i(x)$$

   What classes does this belong to?

2. Random aspiration levels, rejection of current strategy if falls short, and then imitation of randomly drawn individual in own population

3. A well-behaved one-dimensional family of growth-rate functions:

$$g_{ih}(x) = \frac{\exp\left[\sigma\tilde{\pi}\left(e_i^h, x_{-i}\right)\right]}{\sum_{k\in S_i} x_{ik} \exp\left[\sigma\tilde{\pi}\left(e_i^k, x_{-i}\right)\right]} - 1$$

   for some $\sigma > 0$

- Regular? What classes does it belong to?

  [Hint: For CM, use Jensen's inequality!]

- Approaches replicator orbits as $\sigma \to 0$:

$$g_{ih}(x) \approx \frac{1 + \sigma\tilde{\pi}\left(e_i^h, x_{-i}\right) - [1 + \sigma\tilde{\pi}(x)]}{1 + \sigma\tilde{\pi}(x)} \to \sigma \cdot \left[\tilde{\pi}\left(e_i^h, x_{-i}\right) - \tilde{\pi}(x)\right]$$

- Approaches "best-reply dynamic" as $\sigma \to +\infty$:

$$\begin{aligned} g_{ih}(x) \quad &\approx \quad \frac{\exp\left[\sigma\tilde{\pi}\left(e_i^h, x_{-i}\right)\right]}{\sum_{k \in \beta_i(x)} x_{ik} \exp\left[\sigma\tilde{\pi}\left(e_i^k, x_{-i}\right)\right]} - 1 \\ &\to \quad \begin{cases} \left(\sum_{k \in \beta_i(x)} x_{ik}\right)^{-1} - 1 & \forall h \in \beta_i(x) \\ -1 & \forall h \notin \beta_i(x) \end{cases} \end{aligned}$$

2.3 Results

Proposition 2.1 (Hofbauer and Weibull, 1996) *If a pure strategy $h \in S_i$ is iteratively strictly dominated (by a pure or mixed strategy), then its population share converges to zero, from any interior initial state and in* **all CPM dynamics**.

Proof sketch:

1. Suppose $k \in S_i$ is strictly dominated by $y_i \in \Delta(S_i)$:

$$\tilde{\pi}_i(y_i, x_{-i}) > \tilde{\pi}_i(e_i^k, x_{-i}) \ \forall x \in \square(S)$$

2. Let $V : int(\square) \to \mathbb{R}$ be defined by

$$V(x) = \sum_{h \in S_i} y_{ih} \ln(x_{ih}) - \ln(x_{ik})$$

3. Then V increases along any CPM dynamic:

$$\dot{V}(x) = \sum_{h\in S_i} \frac{\partial V(x)}{\partial x_{ih}} \dot{x}_{ih} = \sum_{h\in S} \frac{y_{ih}\dot{x}_{ih}}{x_{ih}} - \frac{\dot{x}_{ik}}{x_{ik}}$$

$$= \sum_{h\in S_i} y_{ih} g_{ih}(x) - g_{ik}(x) = y_i \cdot g_i\,(x) - g_{ik}(x) > 0$$

4. Indeed, one can show that $\exists\delta > 0$ s.t. $\dot{V}\,(x) > \delta\ \forall x$

5. Hence $V(x) \to +\infty$ and thus $x_{ik} \to 0$

- We also show that the result is essentially sharp: if a dynamic is not CPM then $\exists$ game with a strictly dominated strategy that survives in an insignificant population share forever.

Proposition 2.2 (Weibull, 1995) *If an interior solution trajectory converges in* **any WPP dynamic***, then its limit point is a Nash equilibrium.*

Proposition 2.3 (Weibull, 1995) *If* $x \in \square(S)$ *is Lyapunov stable in any* **WPP dynamic***, then* $x \in \square^{NE}$*.*

2.4 Set-wise stability

- For each player role i, let $T_i \subset S_i$ and consider the sub-polyhedron

$$\square(T) = \times_{i \in N} \Delta(T_i)$$

Definition 2.2 $X = \square(T)$ **contains its** *(pure weakly)* **better replies** *if for all $x \in X$:*

$$\tilde{\pi}_i(e_i^h, x_{-i}) \geq \tilde{\pi}_i(x) \;\Rightarrow\; h \in T_i$$

Definition 2.3 *A closed invariant set X is* **asymptotically stable** *if it is Lyapunov stable and has a nbd from which all solution trajectories converge to the set.*

Proposition 2.4 (Ritzberger and Weibull, 1995) *If a subpolyhedron $\square(T)$ contains its better replies, then it is asymptotically stable in all PP dynamics. If a subpolyhedron $\square(T)$ is asymptotically stable in some PP dynamic, then it contains its better replies.*

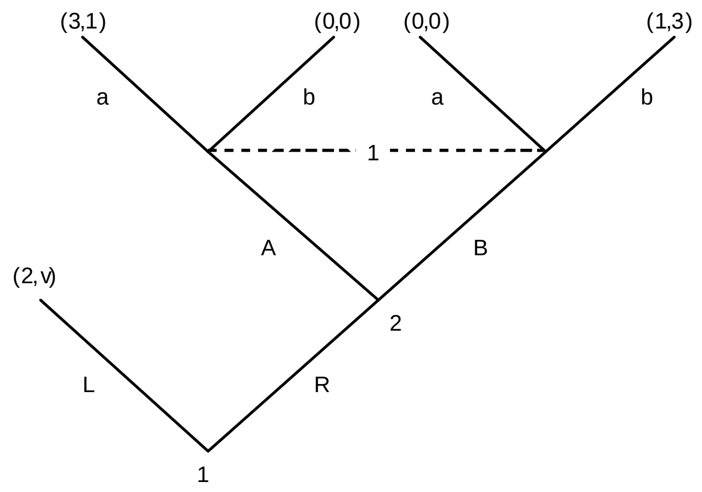
Proposition 2.5 (Ritzberger and Weibull, 1995) *If a subpolyhedron* $\square(T) = \times_{i \in N} \Delta(T_i)$ *contains its better replies, then it contains an essential NE-component and a (KM) strategically stable set, and a proper equilibrium.*

Remark 2.1 *Sufficient for this result that* $\square(T) = \times_{i \in N} \Delta(T_i)$ *contains its* **best** *replies. Such as set is called a* curb *set (closed under rational behavior):*

$$\beta_i[\square(T)] \subset T_i \quad \forall i \in N$$

2.5 Examples

- Reconsider the battle-of-the sexes game where player 1 has an outside option (go to a café with a friend).



- Multiple SPE but unique forward-induction solution: $s^* = (Ra, A)$

- What strategy profiles, or subpolyhedra, are asymptotically stable in PP dynamics?

	A	B
La	$2, v$	$2, v$
Lb	$2, v$	$2, v$
Ra	$3, 1$	$0, 0$
Rb	$0, 0$	$1, 3$

3 Stochastic selection dynamics

[Benaïm and Weibull (2003) and (2009)]

- Finite n-player game $G = (I, S, \pi)$ in normal form [Note change of notation: I instead of N]

- One population, of finite size N, for each player role

- All individuals play pure strategies

- Random draw of 1 individual for *strategy review*, at times $t = 0$, $1/N$, $2/N$...

- Equal probability for each of the nN individuals to be drawn

- *Population state*: vector $X(t) = \langle X_1(t), ..., X_n(t) \rangle$ of population-share vectors $X_i(t) = (X_{ih}(t))_{h \in S_i}$ where $X_{ih}(t) = N_{ih}(t)/N$

- Define a Markov chain $X^N = \left\langle X^N(t) \right\rangle$ on

$$\Theta^N = \{x \in \square(S) : Nx_{ih} \in \mathbb{N} \quad \forall i \in I, h \in S_i\}$$

as follows:

1. $\forall i \in I$ and $h, k \in S_i$ $\exists$ a continuous function $p_{ik}^{hN} : \square(S) \to [0,1]$ s.t.

$$x_{ik} = 0 \;\Rightarrow\; p_{ik}^{hN}(x) = 0$$

and $\forall x \in \Theta^N$:

$$\Pr\left[X_i^N(t + \frac{1}{N}) = x_i + \frac{1}{N}\left(e_i^h - e_i^k\right) \mid X^N(t) = x\right] = p_{ik}^{hN}(x)$$

2. $\forall x \in \Theta^N, y \in \mathbb{R}^m$ $(m = |S_1| + ... + |S_1|)$:

$$\Pr\left[X^N(t+\frac{1}{N}) = x + \frac{1}{N}y \mid X^N(t) = x\right]$$
$$= \begin{cases} p_{ik}^{hN}(x) & \text{if } y_i = e_i^h - e_i^k \text{ and } y_j = 0 \ \forall j \neq i \\ 0 & \text{otherwise} \end{cases}$$

3. Then the *expected net increase* in subpopulation (i, h), from t to $t + 1/N$, conditional upon the current state x, is

$$F_{ih}^N(x) = \sum_{k \neq h} p_{ik}^{hN}(x) - \sum_{k \neq h} p_{ih}^{kN}(x) \ .$$

4. Assume that

   (a) F^N bounded

   (b) $\forall$ compact set C $\exists$ common Lipschitz constant $\forall F^N$

   (c) $F^N \to F$ uniformly

5. Then also F is bounded and locally Lipschitz continuous

6. We are interested in deterministic continuous-time approximation of X^N when N is large

3.1 Mean-field equations

The system of *mean-field* equations:

$$\dot{x}_{ih} = F_{ih}^N(x) \qquad \forall i, h, x, N$$

- *Solution mapping* $\xi^N : \mathbb{R} \times \Box(S) \to \Box(S)$

- The *flow* induced by F^N

- Affine interpolation of the process X^N: $\hat{X}^N$ (connect the points by straight-line segments)

- Deviation between the flow ξ^N and $\hat{X}^N$ at any time $t \in \mathbb{R}$:

$$||\hat{X}^N(t) - \xi^N(t,x)||_\infty = \max_{i \in I, h \in S_i} \left| \hat{X}_{ih}^N(t) - \xi_{ih}^N(t,x) \right|$$

- Given $T < +\infty$, the *maximal deviation* in $[0, T]$:

$$D_N^N(T,x) = \max_{0 \le t \le T} ||\hat{X}^N(t) - \xi^N(t,x)||_\infty$$

- Consider also the limit flow $\xi : \mathbb{R} \times \square(S) \to \square(S)$ that solves

$$\dot{x}_{ih} = F_{ih}(x) \quad \forall i, h, x$$

- Let

$$D^N(T,x) = \max_{0 \le t \le T} ||\hat{X}^N(t) - \xi(t,x)||_\infty$$

Proposition 3.1 $\forall T > 0 \; \exists \, c > 0$ *such that* $\forall \varepsilon > 0$ *and any* N *large enough:*

$$\Pr\left[D^N(T,x) \ge \varepsilon \mid X^N(0) = x\right] \le 2Me^{-\varepsilon^2 cN} \qquad \forall x \in \square^N(S).$$

- Here $M = m - n$, the dimension of the tangent space of $\square(S)$

- This result can be used, in combination with the Borel-Cantelli Lemma, to establish result that connect the behavior of X^N for N large, with properties of its mean-field.

3.2 Exit times

Definition 3.1 *First exit time from a set* $B \subset \square(S)$*:*

$$\tau^N(B) = \mathrm{inf}\left\{t \geq 0 : \hat{X}^N(t) \notin B\right\} \ .$$

- Consider the *forward orbit* $\gamma^+(x^0)$ of the mean-field solution ξ through $x^0 \in \square(S)$

Proposition 3.2 *Let* B *be an open nbd of the closure of* $\gamma^+(x^0)$ *and suppose that* $X^N(0) \to x^0$*. Then*

$$\mathrm{Pr}\left[\lim_{N\to\infty} \tau^N(B) = +\infty\right] = 1$$

- In particular, if the mean-field growth-rate function is WPP, $x^0 \in int\,[\square(S)]$ and $\xi\left(t, x^0\right) \to x^*$, then we know from the above that

$x^* \in \square^{NE}(S)$, and then this proposition says that $X^N(t)$, for large enough N, will stay close to the trajectory of ξ and will remain for a very long time near the Nash equilibrium.

Definition 3.2 *The* **basin of attraction** *of a closed asymptotically stable set $A \subset \square(S)$ is the set*

$$\mathcal{B}(A) = \{x \in \square(S) : \xi(t,x)_{t\to\infty} \to A\}$$

Proposition 3.3 *Let $A \subset \square(S)$ be closed and asymptotically stable set in the mean-field flow ξ. Every nbd $B_1 \subset \mathcal{B}(A)$ of A contains a nbd B_0 of A s.t.*

$$X^N(0) \in B_0 \;\forall N \quad \Rightarrow \quad \mathrm{Pr}\left[\lim \inf_{N\to\infty} \tau^N(B_0) = +\infty\right] = 1$$

- In particular, if the mean-field growth-rate function is PP and $A = \square(T)$ is a subpolyhedron that contains its better replies, then we know from the above that A is asymptotically stable, and thus, for all N large enough: if X^N starts in A, or near it, then X^N will stay in or near A, for a very long time.

3.3 Visitation rates

Definition 3.3 *Let $x \in \square(S)$. A state $y \in \square(S)$ belongs to the* **omega limit set** $\omega(x)$ *of x if* $\lim_{t_k \to +\infty} \xi(t_k, x) = y$ *for some sequence $t_k \to +\infty$.*

Definition 3.4 *The* **Birkhoff center** $B^C(\xi)$ *of a flow ξ is the closure of set of states $x \in \omega(x)$.*

- All stationary states and all points on periodic orbits belong the Birkhoff center.

Definition 3.5 *The* **visitation rate** *in a set* $C \subset \square(S)$ *during a time interval* $[0, T]$ *is*

$$V^N(C,T) = \frac{1}{|\mathbb{T}(T)|} \sum_{t \in \mathbb{T}(T)} 1_{\{X^N(t) \in C\}}$$

where $\mathbb{T}(T)$ *is the set of times* $t = 0, 1/n, 2/N, ... \leq T$.

Proposition 3.4 *For any open nbd* A *of* $B^C(\xi)$:

$$\lim_{N \to \infty} \left[\liminf_{T \to \infty} V^N(A,T) \right] = 1 \quad \textit{a.s.}$$

- In other words: almost all of the time the stochastic process will "hang around" the Birkhoff center of its mean-field.

- Next lecture: stochastic models of noisy best-reply adaptation

- References: Young (1993), Hurkens (1995) and Young (1998)

THE END