On Phase Transition and Percolation in the Beach Model

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Abstract

The beach model, which was introduced by Burton and Steif, has many features in common with the Ising model. We generalize some results for the Ising model to the beach model, such as the connection between phase transition and a certain percolation event. The Potts model extends the Ising model to more than two spin states, and we go on to study the corresponding extension of the beach model. Using random-cluster model methods we obtain some results on where in the parameter space this model exhibits phase transition. Finally we study the beach model on regular trees. Critical values are estimated with iterative numerical methods. In different parameter regions we will see indications of both first and second order phase transition.

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1 Introduction

The beach model was introduced in 1994 by Burton and Steif, see [4, 5]. It is well known that a strongly irreducible subshift of finite type in one dimension has a unique measure of maximal entropy, [21]. The beach model was brought forth as a counterexample to this in higher dimensions. Burton and Steif showed that in some part of the parameter space the model has more than one measure of maximal entropy, called *phase transition* by analogy with the language of statistical mechanics.

The beach model was then somewhat enlarged and further studied by Häggström, [11]. It was shown in [11] that the phenomenon of phase transition was monotone in the model parameter, thus proving the existence of a critical value above which there are multiple measures of maximal entropy and below which there is only one such measure. This is similar to the critical inverse temperature of the Ising model, and its region of phase transition. The main purpose of this paper is to look for such similarities between the beach model and the Ising model. In [23] Wallerstedt examines and shows some other similarities between the two models, such as the global Markov property for the plus measure and certain large deviation properties. See also Häggström [14] for some other results in the same spirit, although in a more general graph context.

The whereabouts of the critical value for the beach model on $\mathbb{Z}^d$ depends on the dimension d. In [11] lower and upper limits were given, resulting in rather broad intervals. However, in [19] Nelander was able to, with a Markov chain Monte Carlo technique, conjecture better estimates for the critical value for low dimensions. Here we will investigate the same question for the beach model on regular trees. The question of phase transition can then be transferred to the question of the number of solutions to a certain fixed point problem. The critical values in the now two-dimensional parameter space are then estimated with iterative numerical methods. In different parameter regions we will see indications of both first and second order phase transition.

The rest of this paper is organized as follows. The general model setting together with the necessary concepts and definitions are presented in Section 2. In Section 3 a short introduction to the Ising model is given, together with some historical notes. In Section 4 the beach model is defined: first in the way it was originally defined by Burton and Steif, to give a clear understanding of its initial purpose, and then in a more general way, thereby reducing the state space, but instead extending the parameter space. In Section 5 the percolation properties of the beach model are compared to those of the Ising model. It is shown that on $\mathbb{Z}^2$, like in the Ising model, phase transition is equivalent to 'plus' percolating in the 'plus measure'. For other (non-planar) graphs, it is shown that 'plus' percolates when there is a phase transition, and that the converse fails. In Section 6 we consider an extension of the beach model analogous to the usual generalization from the Ising model to the Potts model. The (Fortuin–Kasteleyn) random-cluster model, modified to the beach model situation, is introduced. With its help the existence of a critical value for the Potts-like beach model is shown. In Section 7 the beach model on regular trees is studied, and here the model can be viewed as having three parameters. We will see strong indications of where the critical values are located.

2 Preliminary definitions

Our object of study originates from physical systems with many particles, located at the sites of a crystal lattice. But initially we will look at a little more general set-up and allow more than just lattices. Let $\mathcal{G}$ be the set of countably infinite, locally finite, connected graphs. Take some graph $G = (V, E) \in \mathcal{G}$, where V is the vertex set and E the edge set. Write $x \sim y$ if the vertices $x, y \in V$ are adjacent. In this case, x and y are also called neighbours and the edge (or bond) between x and y is denoted by $\langle xy \rangle$. One important example is the case $V = \mathbb{Z}^d$ (the d-dimensional cubic lattice) with edges drawn between sites of unit distance; hence $x \sim y$ whenever $|x - y| = 1$. Here $|\cdot|$ stands for the L^1-norm, i.e. $|x| = \sum_{i=1}^d |x_i|$ whenever $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. This choice is natural because then $|\cdot|$ coincides with the graph-theoretical distance. Such graphs on $\mathbb{Z}^d$ are denoted $(\mathbb{Z}^d, \sim)$.

A region of the sites (vertices), i.e. a subset $\Lambda \subset V$, is called finite if its cardinality $|\Lambda|$ is finite. The complement of a finite region Λ will be denoted by $\Lambda^c = V \setminus \Lambda$. The boundary $\partial\Lambda$ of Λ is the set of all sites in Λ^c which are adjacent to some site of Λ, i.e. $\partial\Lambda = \{x \in \Lambda^c : \exists\, y \in \Lambda \text{ such that } x \sim y\}$.

Let S be a non-empty set called the *state space*. Typically each site will be assigned a value from S. A *configuration* is a map $\sigma : V \to S$, which to each vertex $x \in V$ assigns a value $\sigma(x) \in S$, and can in a magnetic set-up be interpreted as the spin of an elementary magnet at x. Sometimes the value $\sigma(x)$ is therefore referred to as the spin at site x. Two configurations are said to agree on a region $\Lambda \subset V$, written as "$\sigma \equiv \eta$ on Λ", if $\sigma(x) = \eta(x)$ for all $x \in \Lambda$. Similarly, we write "$\sigma \equiv \eta$ off Λ" if $\sigma(x) = \eta(x)$ for all $x \in \Lambda^c$. We also consider configurations in finite regions $\Lambda \subset V$. A configuration $\sigma : \Lambda \to S$ is a *restriction* of a configuration $\eta : \Delta \to S$ if $\Lambda \subset \Delta$ and $\sigma \equiv \eta$ on Λ. We also say that in this case that η is an *extension* of σ.

The space of all configurations, called the configuration space, is the product space $\Omega = S^V$. We equip Ω with the natural underlying σ-field $\mathcal{F} = \sigma(\text{cylinder sets of } \Omega)$. As the spins of the system are supposed to be random, we will consider suitable probability measures μ on $(\Omega, \mathcal{F})$. Each such μ is called a *random field*. Equivalently, the family $X = (X(x), x \in V)$ of random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ which describe the spins at all sites is called a random field.

Definition 2.1 The random object X (or the measure μ) is said to be a *Markov random field* if μ admits conditional probabilities such that for all finite $\Lambda \subset V$, all $\sigma \in S^\Lambda$, and all $\eta \in S^{\Lambda^c}$ we have

$$\mu(X(\Lambda) = \sigma | X(\Lambda^c) = \eta) = \mu(X(\Lambda) = \sigma | X(\partial\Lambda) = \eta(\partial\Lambda)). \tag{1}$$

In other words, the Markov random field property says that the conditional distribution of what we see on Λ, given everything else, only depends on what we see on the boundary $\partial\Lambda$.

A real function $f : \Omega \to \mathbb{R}$ is called *local* if it depends only on finitely many spins. For such functions, let $\|\cdot\|$ denote the supremum norm $\|f\| = \sup_\sigma |f(\sigma)|$.

2.1 Stochastic domination

Suppose that S is a subset of $\mathbb{R}$, so that the elements of S are ordered. The configurations space Ω is then equipped with a natural partial order $\preceq$ which is defined coordinate-wise: For $\sigma, \sigma' \in \Omega$, we write $\sigma \preceq \sigma'$ (or $\sigma' \succeq \sigma$) if $\sigma(x) \leq \sigma'(x)$ for every $x \in V$. A function $f : \Omega \to \mathbb{R}$ is said to be *increasing* if $f(\sigma) \leq f(\sigma')$ whenever $\sigma \preceq \sigma'$. An event A is said to be increasing if its indicator function $\mathbf{1}_A$ is increasing. The following standard definition of stochastic domination expresses the fact that μ' prefers larger elements of Ω than μ.

Definition 2.2 Let μ and μ' be two probability measures on Ω. We say that μ is *stochastically dominated* by μ', or μ' is stochastically larger than μ, writing $\mu \preceq_{\mathcal{D}} \mu'$, if for every bounded increasing function $f : \Omega \to \mathbb{R}$ we have $\mu(f) \leq \mu'(f)$.

The following fundamental result of Strassen [22] characterizes stochastic domination in coupling terms.

Theorem 2.3 (Strassen) *For any two probability measures μ and μ' on Ω, the following statements are equivalent.*

(i) $\mu \preceq_{\mathcal{D}} \mu'$

(ii) *For all* continuous *bounded increasing functions $f : \Omega \to \mathbb{R}$, $\mu(f) \leq \mu'(f)$.*

(iii) *There exists a coupling P of μ and μ' such that $P(X \preceq X') = 1$.*

The equivalence (i) $\Leftrightarrow$ (ii) in Theorem 2.3 implies that the relation $\preceq_{\mathcal{D}}$ of stochastic domination is preserved under weak limits.

Next we recall a sufficient condition for stochastic domination. This condition is essentially due to Holley [15] and refers to the finite-dimensional case when $|V| < \infty$. We also assume for simplicity that $S \subset \mathbb{R}$ is finite. Hence Ω is finite. In this case, we call a probability measure μ on Ω *connected* if, for any $\sigma, \eta \in \Omega$ such that both σ and η have positive μ-probability, we can move from σ to η through single-site changes without passing through any element of zero μ-probability.

Theorem 2.4 (Holley) *Let X and X' be Ω-valued random elements with connected distributions μ and μ', and assume that μ' assigns positive probability to the maximal element of Ω. If for all $x \in V$, all $s \in S$, μ-a.a. $\sigma \in S^{V\setminus\{x\}}$ and μ'-a.a. $\eta \in S^{V\setminus\{x\}}$ such that $\sigma \preceq \eta$ we have*

$$\mu(X(x) \geq s \,|\, X(V \setminus \{x\}) = \sigma) \leq \mu'(X'(x) \geq s \,|\, X'(V \setminus \{x\}) = \eta),$$

then $\mu \preceq_{\mathcal{D}} \mu'$.

Definition 2.5 A probability measure μ on Ω is said to have *positive correlations* if for all bounded increasing functions $f, g : \Omega \to \mathbb{R}$ we have

$$\mu(fg) \geq \mu(f)\mu(g). \tag{2}$$

More or less as a corollary to Holley's Theorem 2.4 we get the well known FKG inequality. See [9] for a proof.

Theorem 2.6 (The FKG inequality) *Let V be finite, S a finite subset of $\mathbb{R}$, and μ a probability measure on Ω which is connected and assigns positive probability to the maximal element of Ω. If μ is monotone, meaning*

$$\mu(X(x) \geq a | X = \xi \text{ off } x) \leq \mu(X(x) \geq a | X = \eta \text{ off } x)$$

whenever $x \in V$, $a \in S$, and $\xi, \eta \in S^{V\setminus\{x\}}$ are such that $\xi \preceq \eta$, $\mu(X = \xi$ off $x) > 0$ and $\mu(X = \eta$ off $x) > 0$, then μ also has positive correlations.

Finally we state a simple observation from [17] used later on. It says that if two probability measures have the same marginal distributions and are comparable in the sense of stochastic domination, then they are in fact equal.

Proposition 2.7 *Let V be finite or countable, and let μ and μ' be two probability measures on $\Omega = S^V$ satisfying $\mu \preceq_{\mathcal{D}} \mu'$. If, in addition, $\mu(X(x) \leq r) = \mu'(X(x) \leq r)$ for all $x \in V$ and $r \in S$ then $\mu = \mu'$.*

Proof. Let P be a coupling of μ and μ' such that $P(X \preceq X') = 1$ which exists by Theorem 2.3. Writing $\mathbb{Q}$ for the set of rational numbers, we have for each $x \in V$

$$\begin{aligned} P(X(x) \neq X'(x)) &= P(X(x) < X'(x)) \leq \sum_{r \in \mathbb{Q}} P(X(x) \leq r, X'(x) > r) \\ &= \sum_{r \in \mathbb{Q}} (P(X(x) \leq r) - P(X'(x) \leq r)) \\ &= 0. \end{aligned}$$

Summing over all $x \in V$ we get $P(X \neq X') = 0$, whence $\mu = \mu'$ by the coupling inequality $\|\mu - \mu'\|_{TV} \leq P(X \neq X')$. $\square$

3 The Ising model

The Ising model on a graph $G = (V, E)$ is a certain random assignment of $+1$'s and -1's to the vertices of G. It was introduced in the 1920s as a model for ferro-magnetism, and is today the most studied of all Markov random field models; see e.g. [8, 16] for introductions and some history. In our setting we have $S = \{-1, 1\}$. A probability measure on $\Omega = S^V$ is said to be a *Gibbs measure for the Ising model on G at inverse temperature $\beta \geq 0$* if it is Markov and for all finite $\Lambda \subset V$ and all $\sigma \in S^{\Lambda}$, $\eta \in S^{\partial\Lambda}$ we have

$$\mu(X(\Lambda) = \sigma | X(\partial\Lambda) = \eta) = \frac{1}{Z} \exp\left[\beta\left(\sum_{\substack{x,\, y \in \Lambda: \\ x \sim y}} \sigma(x)\sigma(y) + \sum_{\substack{x \in \Lambda,\, y \in \partial\Lambda: \\ x \sim y}} \sigma(x)\eta(y)\right)\right]$$

Here Z is a normalizing constant which depends on β, Λ and η but not on σ. For $\beta = 0$ ("infinite temperature") the spin variables are independent under μ, but as soon as $\beta > 0$ the probability distribution starts to favour configurations with many neighbour pairs of aligned spins. This tendency becomes stronger and stronger as β increases.

The existence of Gibbs measures on $(\mathbb{Z}^d, \sim)$ can be established using stochastic domination of Gibbs measures on an increasing sequence of finite regions growing to $\mathbb{Z}^d$ with suitable boundary conditions. It is well known that the existence of more than one Gibbs measure, called *phase transition* by analogy with the language of statistical mechanics, is increasing in β. This was originally proved using so-called Griffiths inequalities (see e.g. [17]); the modern approach is based on the random-cluster model, see [13]. The following result is an immediate consequence.

Theorem 3.1 *For the Ising model on the integer lattice $\mathbb{Z}^d$ of dimension $d \geq 2$ there exists a critical inverse temperature $\beta_c \in [0, \infty)$ (depending on d) such that for $\beta < \beta_c$ the model has a unique Gibbs measure while for $\beta > \beta_c$ there are multiple Gibbs measures.*

For $\mathbb{Z}^2$ the critical value has been found to be $\beta_c = \frac{1}{2}\log(1 + \sqrt{2})$, see [20]. Later it was also shown in [1] that the model has a unique Gibbs measure *at* the critical value $\beta = \beta_c$. For higher dimensions a rigorous calculation of the critical value is beyond current knowledge. It is believed that uniqueness holds at criticality in all dimensions $d \geq 2$, but so far this is only known for $d = 2$ and $d \geq 4$, see [2].

Phase transition can in two dimensions be completely characterized by the following percolation phenomenon. The result is due to Coniglio et al. [7].

Theorem 3.2 *For the Ising model on the square lattice $\mathbb{Z}^2$ at inverse temperature β, the μ_β^+-probability of having an infinite plus-cluster is 0 in the uniqueness regime $\beta \leq \beta_c$, and 1 in the non-uniqueness regime $\beta > \beta_c$.*

Here μ_β^+ is the limiting Gibbs measure obtained by letting all boxes $\Lambda_n = [-n, n] \times [-n, n]$ in the sequence $(\Lambda_n)_{n \geq 1}$ have a complete $+1$-boundary.

We shall see that the beach model also possesses this equivalence of non-uniqueness and percolation in two dimensions. For $d \geq 3$, or for non-planar graphs in general, the corresponding sharp equivalence is no longer true in either model. This will be shown in Section 5.

4 The beach model

The beach model was introduced by Burton and Steif, see [4] and [5], as an example of a strongly irreducible subshift of finite type, which has for some choice of the model parameter more than one measure of maximal entropy. The model was somewhat enlarged and further studied by Häggström, [11]. It was shown in [11] that the parameter has a critical value above which there is more than one measure of maximal entropy and below which there is only one such measure. This is similar to the phase transition phenomenon for the Ising model. The whereabouts of this critical value in the $\mathbb{Z}^d$ case could only be given as a rather broad interval, with endpoints depending on the dimension d.

4.1 The model as a subshift of finite type

An *automorphism* of a graph G is a bijective mapping $\gamma : V \to V$ such that $x \sim y \Leftrightarrow \gamma x \sim \gamma y$. Assume for now that G is a transitive graph, i.e. for any $x, y \in V$ there exists an automorphism γ such that $\gamma x = y$. Each such automorphism also induces a transformation of the configuration space Ω. One important class of automorphisms are the translations of the integer lattice $V = \mathbb{Z}^d$, $\gamma_y x = x + y, y \in \mathbb{Z}^d$. The associated translation group acting on Ω is then given by $T_y\sigma(x) = \sigma(\gamma_y x) = \sigma(x+y)$, $x, y \in \mathbb{Z}^d$. In particular, any constant configuration is translation invariant. Similarly, we can speak of periodic configurations which are invariant under T_y.

Definition 4.1 Let $\sigma_i : \Gamma_i \to S, 1 \leq i \leq K$, be a finite set H of configurations with $\Gamma_i \subset \mathbb{Z}^d$ finite for each $1 \leq i \leq K$. The *subshift of finite type* (in d dimensions) corresponding to H is the set $\mathbf{X} \subset S^{\mathbb{Z}^d}$ consisting of all configurations $\sigma : \mathbb{Z}^d \to S$ such that for all $y \in \mathbb{Z}^d$, it is not the case that $T_y\sigma$ is an extension of some σ_i (The σ_i's should be thought of as the disallowed finite configurations).

Subshifts of finite type (SOFTs) are shift invariant, i.e. $\sigma \in \mathbf{X}$ and $y \in \mathbb{Z}^d$ implies $T_y\sigma \in \mathbf{X}$.

A configuration $\tilde{\sigma} : \Gamma \to S$ is said to be *compatible* (with $\mathbf{X}$) if $\exists\, \sigma \in \mathbf{X}$ such that $\tilde{\sigma}$ is a restriction of σ.

Definition 4.2 Let $\mathbf{X}$ be a SOFT. $\mathbf{X}$ is *strongly irreducible* if there is an $r \geq 0$ such that whenever we have two finite compatible configurations $\sigma_1 : \Gamma_1 \to S$ and $\sigma_2 : \Gamma_2 \to S$ and the distance between Γ_1 and Γ_2 is greater than r, there is an $\sigma \in \mathbf{X}$ that is an extension of both σ_1 and σ_2.

The next definition gives a measure of the degree of complexity of a SOFT. Let $\Lambda_n = [-n, n]^d$ and $\mathbf{X}_n = \{\tilde{\sigma} : \Lambda_n \to S \text{ with } \tilde{\sigma} \text{ compatible}\}$. Further we let $N_n = |\mathbf{X}_n|$ and finally $\mathbf{X}(\tilde{\sigma}) = \{\sigma \in \mathbf{X} : \sigma \text{ is an extension of } \tilde{\sigma}\}$.

Definition 4.3 The *topological entropy* of $\mathbf{X}$ is

$$H(\mathbf{X}) = \lim_{n\to\infty} \frac{\log N_n}{|\Lambda_n|}.$$

Suppose that μ is a translation invariant probability measure on $\mathbf{X}$. Then the *measure theoretic entropy* of μ is

$$H(\mu) = \lim_{n\to\infty} -\frac{1}{|\Lambda_n|} \sum_{\tilde{\sigma}\in\mathbf{X}_n} \mu(\mathbf{X}(\tilde{\sigma})) \log \mu(\mathbf{X}(\tilde{\sigma})).$$

Both of these limits exist by sub-additivity. Clearly for any such μ we have $H(\mu) \leq H(\mathbf{X})$. It is in fact well known that $H(\mathbf{X}) = \sup_\mu H(\mu)$ where the supremum is taken over all translation invariant probability measures on $\mathbf{X}$. Moreover, the supremum is achieved at some measure (see [18]). In the case of strong irreducibility there is in 1 dimension always a unique measure of maximal entropy. However, for $d \geq 2$ there sometimes exist more than one measure of maximal entropy. By analogy with the Ising model we say that we have a *phase transition* when multiple measures of maximal entropy exist. Here it will be exemplified by the beach model.

The following characterization of measures with maximal entropy for strongly irreducible SOFTs is from [4].

Proposition 4.4 *Let* $\mathbf{X}$ *be a strongly irreducible SOFT for which the disallowed finite configurations consists only of pairs of neighbours. Let* μ *be a translation invariant probability measure on* $\mathbf{X}$*. Then the following statements are equivalent.*

(i) μ *is a measure of maximal entropy.*

(ii) *The conditional distribution of* μ *on any finite set* Γ *given the configuration on* $\partial\Gamma$ *is* μ*-a.s. uniform over all configurations on* Γ *which (together with configuration on* $\partial\Gamma$*) extend the configuration on* $\partial\Gamma$*.*

We are now ready to describe the beach model.

Definition 4.5 Take $d \geq 2$, $M \in \{2, 3, \ldots\}$ and let $S' = \{-M, -M+1, \ldots, -1, 1, \ldots, M-1, M\}$. The *beach model in d dimensions with parameter M* is the d-dimensional SOFT where a negative in S' may not sit next to a positive, unless they both have one as their absolute value.

It is the interpretation in two dimensions of the symbols representing altitude above the sea level that has given rise to the name of the model; the rules of the model prevent the shores from being too steep.

The beach model is a SOFT satisfying the conditions of Proposition 4.4. This tells us that for the beach model, looking for measures of maximal entropy, is the same as looking for measures with uniform conditional distributions. So assume μ satisfies this condition and it is now a question of existence and uniqueness of such a μ. In [4] Burton and Steif showed that in $d \geq 2$ dimensions the beach model exhibits phase transition if the parameter M is large enough:

Proposition 4.6 *Consider the beach model and let* $d \geq 2$*. If*

$$M > 4e28^d$$

then there are exactly 2 ergodic measures of maximal entropy in d dimensions.

4.2 The model with a continuous parameter

In the above setting, the parameter of the model, M, is integer-valued. To extend the parameter space, reduce the state space S' to $S = \{-2, -1, 1, 2\}$ where the former spins $2, 3, \ldots, M$ now are represented by the 2 and similarly for the negative spins. To keep what once were uniform conditional distributions, we let the measure put weights $(M-1), 1, 1, (M-1)$ respectively on the new spins. The model parameter M can now take all real values in $(1, \infty)$.

Consider some graph $G = (V, E) \in \mathcal{G}$. As before, we say that a configuration $\sigma \in \{-2, -1, 1, 2\}^\Lambda$ with $\Lambda \subset V$ is *BM-feasible* if for each neighbour pair $x \sim y$ we have $\sigma(x)\sigma(y) \geq -1$.

Definition 4.7 A probability measure μ on S^V is said to be a *Gibbs measure for the beach model on* $G = (V, E)$ *with parameter* $M > 1$ if for all finite $\Lambda \subset V$, all $\sigma \in S^\Lambda$ and μ-a.a. $\eta \in S^{\Lambda^c}$ we have

$$\mu(X(\Lambda) = \sigma | X(\Lambda^c) = \eta) = \frac{1}{Z}(M-1)^{n_{-2}(\sigma)+n_{+2}(\sigma)} \mathbf{1}_{\{(\sigma \vee \eta) \text{ BM-feasible}\}}. \quad (3)$$

Here $n_{-2}(\sigma)$ and $n_{+2}(\sigma)$ are the number of -2's and $+2$'s in σ. The reason why we use the quantifier 'μ-a.a.' rather than 'all BM-feasible' for the set of outer configurations is that certain BM-feasible configurations η may cause $(\sigma \vee \eta)$ to be not BM-feasible for every $\sigma \in S^\Lambda$. Note that a beach model Gibbs measure, conditioned on the configuration η outside Λ, only depends on η through the condition that $(\sigma \vee \eta)$ should be BM-feasible, and that this in turn only involves the configuration on the region boundary $\partial\Lambda$. For that reason it is sometimes more convenient to simply condition on $\{X(\partial\Lambda) = \eta(\partial\Lambda)\}$ rather than on $\{X(\Lambda^c) = \eta(\Lambda^c)\}$, getting the same conditioned measure. To conclude, a beach model Gibbs measure has the Markov random field property (1).

Next we construct Gibbs measures for the beach model on G. The following lemma is useful. Let $\Lambda \subset V$ be finite and for any $\eta \in S^{\Lambda^c}$, write $\mu_{\Lambda,\eta}$ for the beach model measure on Ω that agrees with η outside Λ and else follows the right-hand side of (3). It could be argued that such a measure should be called $\mu^M_{\Lambda,\eta}$ to stress the fact that the parameter for the measure is M, but we will suppress this dependence on M.

Lemma 4.8 *Let* $\Lambda \subset V$ *be finite, and let* η_1 *and* η_2 *be two spin configurations on* Λ^c *satisfying* $\eta_1 \preceq \eta_2$. *Then we have*

$$\mu_{\Lambda,\eta_1} \preceq_{\mathcal{D}} \mu_{\Lambda,\eta_2}.$$

Proof. (Sketch) The idea is to use Holley's Theorem 2.4. We need to check for all $x \in \Lambda$, all $s \in \{-2, -1, 1, 2\}$ and all $\eta \in S^{\Lambda^c}$ that $\mu_{\Lambda,\eta}(X(x) \geq s | X(\Lambda \backslash \{x\}) = \sigma)$ is increasing in σ. But this could easily be done. Here is an example when $s = 2$.

$$\mu_{\Lambda,\eta}(X(x) \geq 2 | X(\Lambda \setminus \{x\}) = \sigma) = \begin{cases} 0 & \text{if } \sigma(\partial\{x\}) \text{ contains } -1 \text{ or } -2, \\ \frac{M-1}{M+1} & \text{if } \sigma(\partial\{x\}) \equiv 1, \\ \frac{M-1}{M} & \text{otherwise} \end{cases}$$

This conditional probability is increasing in σ, and so are the similar expressions for the cases $s = -2, -1, 1$. The claim in the lemma follows. $\square$

Let $(\Lambda_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of V converging to V in the sense that each $x \in V$ is in all but finitely many of the Λ_n's. We refer to such a sequence as an exhaustion of G. Fix a vertex $o \in \Lambda_1$ called the origin.

Let $\mu_{+,n}$ be the probability measure on S^V corresponding to taking $X(\Lambda_n^c) \equiv 2$ and picking $X(\Lambda_n)$ according to (3) with $\Lambda = \Lambda_n$ and $\eta \equiv 2$. Then these measures are stochastically ordered:

$$\mu_{+,1} \succeq_{\mathcal{D}} \mu_{+,2} \succeq_{\mathcal{D}} \cdots . \tag{4}$$

This follows from Lemma 4.8 because $\mu_{+,n}$ could be obtained from $\mu_{+,n+1}$ by conditioning on the increasing event that $X \equiv 2$ on $\Lambda_{n+1} \setminus \Lambda_n$.

We see (by compactness of $\{-2,-1,1,2\}^V$) that the sequence $(\mu_{+,n})_{n=1}^{\infty}$ has a limit. This limit, called the 'plus measure', is denoted μ_+. To see that this is a Gibbs measure for the beach model, we need to check for any finite $\Lambda \subset V$ and any $\eta \in S^{\partial\Lambda}$ that μ_+ satisfies (3). This however, is immediate from the fact that the same property holds for $\mu_{+,n}$ for each n which is large enough for $\Lambda \cup \partial\Lambda$ to be contained in Λ_n.

The limiting measure μ_+ is independent of the choice of exhaustion. Assume $(\Lambda')_{n=1}^{\infty}$ is another exhaustion of G with measures $\mu'_{+,n}$ and limit μ'_+, and we will see that we must have $\mu_+ = \mu'_+$. We are done if it can be established that $\mu_+ \preceq_{\mathcal{D}} \mu'_+$, because then by symmetry $\mu'_+ \preceq_{\mathcal{D}} \mu_+$ implying $\mu_+ = \mu'_+$. In fact, it is enough to show $\mu_+ \preceq_{\mathcal{D}} \mu'_{+,n}$, since stochastic domination is preserved under weak limits. So fix n and find m_0 big enough so that $\Lambda_m \supset \Lambda'_n$ for all $m > m_0$. Then $\mu_{+,m} \preceq_{\mathcal{D}} \mu'_{+,n}$, which yields the desired domination upon letting $m \to \infty$. The result is summarized as follows.

Proposition 4.9 *The limiting probability measure*

$$\mu_+ = \lim_{n\to\infty} \mu_{+,n}$$

on $\{-2,-1,1,2\}^V$ exists and is a Gibbs measure for the beach model on G. The limit is independent of the choice of exhaustion.

By the symmetry of the model, we of course have a measure analogous to μ_+, the 'minus measure' μ_-, obtained with boundary condition -2 rather than 2. Now, from Lemma 4.8 we have

$$\mu_{-,n} \preceq_{\mathcal{D}} \mu_{\Lambda_n,\eta} \preceq_{\mathcal{D}} \mu_{+,n} \tag{5}$$

for any $\eta \in \Omega$ and $n \in \mathbb{N}$. Let μ be any Gibbs measure for the beach model with parameter M. Taking the average $\int \mu(d\eta)$ in (5), we obtain $\mu_{-,n} \preceq_{\mathcal{D}} \mu \preceq_{\mathcal{D}} \mu_{+,n}$, and in the limit

$$\mu_- \preceq_{\mathcal{D}} \mu \preceq_{\mathcal{D}} \mu_+ \tag{6}$$

where μ is any beach model Gibbs measure. Here we see that μ_- and μ_+ play a special role and are extreme in the sense of stochastic ordering.

By *tail triviality* of a measure μ on $(\Omega, \mathcal{F})$ we mean the following. For some ordering of the vertices $V = \{v_1, v_2, \ldots\}$, let $\mathcal{F}_n = \sigma(X(v_{n+1}), X(v_{n+2}), \ldots)$ and let $\mathcal{T} = \cap_n \mathcal{F}_n$. In words, $\mathcal{T}$ is the collection of events that do not alter when changing a finite number of spins. μ is said to have trivial tail if for all events $A \in \mathcal{T}$, either $\mu(A) = 0$ or $\mu(A) = 1$. To check that $\mathcal{T}$ does not depend on the vertex order, let $v'_1, v'_2, \ldots$ be some other ordering and let π denote the permutation defined by $v'_{\pi(n)} = v_n$, $n \in \mathbb{N}$. If $\mathcal{F}'_n = \sigma(X(v'_{n+1}), X(v'_{n+2}), \ldots)$ and $\mathcal{T}' = \cap_n \mathcal{F}'_n$, then we will show $\mathcal{T} = \mathcal{T}'$. Take some event $A \notin \mathcal{T}$. Then there exist some m such that $A \notin \mathcal{F}_m$. Define $M = \max(\pi(1), \ldots, \pi(m))$. Then $\{v_1, \ldots, v_m\} \subset \{v'_1, \ldots, v'_M\}$ and therefore $\mathcal{F}'_M \subset \mathcal{F}_m$. Thus $A \notin \mathcal{F}'_M$ and obviously $A \notin \mathcal{T}'$. This shows $\mathcal{T}' \subset \mathcal{T}$ and by symmetry $\mathcal{T} = \mathcal{T}'$ follows.

The following properties of μ_+ will be useful later on.

Proposition 4.10 *The measure μ_+ has positive correlations.*

Proof. Since the inequality (2) is preserved under rescaling and addition of constants to f and g, μ has positive correlations whenever $\mu \preceq_{\mathcal{D}} \mu'$ for any probability measure μ' with bounded increasing Radon-Nikodym density relative to μ. Theorem 2.3 thus shows that μ has positive correlations whenever $\mu(fg) \geq \mu(f)\mu(g)$ for all *continuous* bounded increasing functions f and g. Hence, the property of positive correlations is also preserved under weak limits. Since $\mu_{+,n}$ has positive correlations from Theorem 2.6 and Lemma 4.8, the claim follows. □

Proposition 4.11 *Let $G \in \mathcal{G}$ be a transitive graph. The measure μ_+ is then automorphism invariant and has trivial tail.*

Proof. From Proposition 4.9 the measure μ_+ is independent of the choice of exhaustion. To any automorphism could the corresponding change of exhaustion be made, thus making it clear that μ_+ is invariant under automorphisms.

To show tail triviality for μ_+ assume, to get a contradiction, that there exists some tail event A with probability $\alpha = \mu(A)$ such that $0 < \alpha < 1$. Let $\mu_1 = \frac{1}{\alpha}\mu_+|_A$ and $\mu_2 = \frac{1}{1-\alpha}\mu_+|_{A^c}$. A moments though reveals that these measures are Gibbs measures as the restriction to a tail event does not influence the conditional probabilities on finite regions. Moreover, we see that μ_+ is a convex combination of the other two Gibbs measures; $\mu_+ = \alpha \cdot \mu_1 + (1-\alpha)\cdot\mu_2$. For any increasing event B we know from (6) that

$$\alpha\mu_1(B) + (1-\alpha)\mu_2(B) \leq (\alpha + 1 - \alpha)\mu_+(B) = \mu_+(B)$$

with equality if, and only if, $\mu_1(B) = \mu_2(B) = \mu_+(B)$. But we already know that equality holds, so substituting B for $\{X(x) \geq r\}$, $x \in V$, $r \in S$ and using Proposition 2.7 gives $\mu_1 = \mu_2 = \mu_+$. We thereby have the desired contradiction, because for example $\mu_1(A) = 1$ while $\mu_+(A) = \alpha < 1$. □

Let $\{X(x) = +\}$ denote the event $\{X(x) \in \{+1, +2\}\}$ for $x \in V$ and $X \in S^V$, and define the event $\{X(x) = -\}$ analogously.

Proposition 4.12 *For the beach model on a graph $G \in \mathcal{G}$ with parameter M, the following statements are equivalent.*

(i) *There is more than one Gibbs measure;*

(ii) $\mu_+ \neq \mu_-$*;*

(iii) $\mu_+(X(o) = +) > \frac{1}{2}$*;*

(iv) $\exists\, \varepsilon > 0$ *such that* $\mu_{+,n}(X(o) = +) \geq \frac{1}{2} + \varepsilon$ *for all* n.

Here we will only show (i)⇔(ii)⇐(iii)⇔(iv), and postpone the missing link (ii)⇒(iii) to Section 6.2.

Proof of (i)⇔(ii)⇐(iii)⇔(iv) in Proposition 4.12. The implications (iv)⇒(iii) and (ii)⇒(i) are immediate. (i)⇒(ii) follows from the relation (6). (iii)⇒(iv) is obvious observing from (4) that $\mu_{+,n}(X(o) = +)$, $n = 1, 2, \ldots$ is a decreasing sequence with a limit $> 1/2$. (iii)⇒(ii): By $\pm$-symmetry $\mu_-(X(o) =$

$+) = \mu_+(X(o) = -) = 1 - \mu_+(X(o) = +) < 1/2$, so μ_- and μ_+ differ at the origin. $\square$

It is also known that the existence of more than one Gibbs measure is increasing in M. The following is an immediate consequence.

Theorem 4.13 *For any graph $G \in \mathcal{G}$ there exists a critical value $M_c = M_c(G) \in [1, \infty]$ such that for $M < M_c$ we have that the beach model on G with parameter M has a unique Gibbs measure whereas for $M > M_c$ there are multiple Gibbs measures.*

A proof using a random-cluster approach can be found in [14], but also in Theorem 6.14 below for a somewhat more general model. For $G = (\mathbb{Z}^d, \sim)$, this result was first obtained by Häggström [11]. There the critical value $M_c(\mathbb{Z}^d)$ was shown to belong to the interval

$$\left(\frac{2d^2 + d + 1}{2d^2 + d - 1}, \exp\{2^{2d-2} \log(1 + \sqrt{2})\} \right). \tag{7}$$

For $d = 2$, this amounts to the interval $(1.222, 33.971)$, and in fact simulations indicate that $M_c(\mathbb{Z}^2) \in (2.1, 2.2)$ see [19].

5 Percolation

Consider again a countably infinite (and locally finite) graph $G = (V, E)$. We designate each vertex a value of either 0 or 1. We call the sites assigned 1 *open* and those assigned 0 *closed*. Let $\Omega_s = \{0,1\}^V$ be the set of configurations and consider a configuration $X \in \Omega_s$, obtained in some random way. We say that there is an *open path* in X from $x \in V$ to $y \in V$ if there is some path from x to y in which all vertices are open. This event is denoted $\{x \leftrightarrow y\}$. We also write $\{x \leftrightarrow \infty\}$ if x is connected to an open path of infinite length.

Bernoulli percolation is a natural way of assigning the open and closed vertices. Each vertex is then open with probability p and closed with probability $1-p$. This is done *independently* for every vertex. The corresponding measure ψ_p on $(\Omega_s, \mathcal{F}_s)$ is thus a product measure:

$$\psi_p = \prod_{x \in V} \pi_x \tag{8}$$

where π_x is given by

$$\pi_x(X(x) = 0) = 1 - p, \quad \pi_x(X(x) = 1) = p$$

for $X \in \Omega_s$. $\mathcal{F}_s$ is the σ-field generated by the finite-dimensional cylinders of Ω_s. The *percolation probability* is defined $\theta(p) = \psi_p(o \leftrightarrow \infty)$. A simple coupling argument shows that $\theta(p)$ is increasing in p and we define the *critical probability* as

$$p_c = \sup\{p : \theta(p) = 0\}. \tag{9}$$

For more on percolation, see [10].

5.1 Agreement percolation

We consider again the general setting of Section 2. Suppose μ is a random field and $\eta \in \Omega$ a fixed configuration. Let $\{x \stackrel{\eta}{\longleftrightarrow} \infty\}$ denote the event that $x \in V$ belongs to an infinite cluster of the random set $R(\eta) = \{y \in V : X(y) = \eta(y)\}$. This idea can also be extended to more than one such fixed configuration η. Let $H = \{\eta_i \in \Omega, i = 1, \dots, N\}$ be a finite set of fixed configurations. As above we consider the event $\{x \stackrel{H}{\longleftrightarrow} \infty\}$ that $x \in V$ belongs to an infinite cluster of the set $R(H) = \{y \in V : X(y) = \eta(y) \text{ for some } \eta \in H\}$. Moreover, we say that μ *exhibits agreement percolation for* H if $\mu(x \stackrel{H}{\longleftrightarrow} \infty) > 0$ for some $x \in V$. To visualize such an agreement, it may be convenient to think of a reduced description of μ in terms of its image under the map $s_H : \Omega \to \Omega_s$, which describes local agreement and disagreement with H, and is defined by

$$(s_H(\sigma))(x) = \begin{cases} 1 & \text{if } \sigma(x) = \eta(x) \text{ for some } \eta \in H, \\ 0 & \text{otherwise} \end{cases}$$

With this mapping, we can write $\{x \stackrel{H}{\longleftrightarrow} \infty\} = s_H^{-1}\{x \leftrightarrow \infty\}$.

5.2 Looking at signs only

We will investigate how agreement percolation is related to phase transition in the beach model. It is natural to consider events like the 'plus sites' percolate.

For this purpose we take $\eta_1 \equiv +1$ and $\eta_2 \equiv +2$ as and form $H^+ = \{\eta_1, \eta_2\}$. Now agreement with H^+ is the same as the considered sites having plus signs. We write $\{x \stackrel{+}{\longleftrightarrow} \infty\}$ for the event that x belongs to an infinite plus cluster, i.e.

$$\{x \stackrel{+}{\longleftrightarrow} \infty\} = \{x \stackrel{H^+}{\longleftrightarrow} \infty\}.$$

For every configuration $\xi \in \Omega$ we can talk about the sign configuration $\eta \in \{0,1\}^V = \Omega_s$ of ξ, by identifying plus sites with 1 and minus sites with 0: $\eta = s_{H^+}(\xi)$. Likewise, any measure μ on $(\Omega, \mathcal{F})$ induces a measure ν on $(\Omega_s, \mathcal{F}_s)$ in the same way:

$$\nu(Y = \eta) = \mu(X \in s_{H^+}^{-1}(\eta)),$$

where $Y \in \Omega_s$ and $X \in \Omega$. The definition for ν is written shorter as $\nu(\eta) = \mu(s_{H^+}^{-1}(\eta))$. In particular, every Gibbs measure for the beach model μ induces a corresponding 'sign measure' $\nu = \mu \circ s_{H^+}^{-1}$.

Lemma 5.1 *Let μ_1 and μ_2 be two measures on $(\Omega, \mathcal{F})$ and let ν_1 and ν_2 be their corresponding induced measures on $(\Omega_s, \mathcal{F}_s)$. Then*

$$\mu_1 \preceq_{\mathcal{D}} \mu_2 \Rightarrow \nu_1 \preceq_{\mathcal{D}} \nu_2.$$

Proof. Assume $\mu_1 \preceq_{\mathcal{D}} \mu_2$. From Strassen's Theorem 2.3 we know there exists some coupling P such that $P(X_1 \preceq X_2) = 1$ and $X_1 \sim \mu_1$, $X_2 \sim \mu_2$. Let $Y_1 = s_{H^+}(X_1)$ and $Y_2 = s_{H^+}(X_2)$. Then, since s_{H^+} preserves order, we have that $P(Y_1 \preceq Y_2) = 1$ and also that $Y_1 \sim \nu_1$, $Y_2 \sim \nu_2$. Using Strassen's Theorem once again we find that $\nu_1 \preceq_{\mathcal{D}} \nu_2$. □

Let $\nu_{+,n}$ be the measure corresponding to $\mu_{+,n}$ for $n \in \mathbb{N}$ and define ν_+ as the measure corresponding to μ_+. From Lemma 5.1 and (4) it follows that

$$\nu_{+,1} \succeq_{\mathcal{D}} \nu_{+,2} \succeq_{\mathcal{D}} \cdots.$$

As before, we see that the sequence $(\nu_{+,n})_{n=1}^{\infty}$ has a limit, call it $\nu_{+,\infty}$. Actually, $\nu_{+,\infty}$ and ν_+ are the same measure as can be seen in the following calculation: For any $A \in \mathcal{F}_s$,

$$\begin{aligned} \nu_{+,\infty}(A) &= \lim_{n\to\infty} (\mu_{+,n} \circ s_{H^+}^{-1})(A) \\ &= \lim_{n\to\infty} \mu_{+,n}(s_{H^+}^{-1}(A)) = \mu_+(s_{H^+}^{-1}(A)) \\ &= \nu_+(A). \end{aligned}$$

Proposition 5.2 *Let $G \in \mathcal{G}$ be a transitive graph. Then ν_+ is automorphism invariant, has positive correlations and has trivial tail.*

Proof. The three properties are inherited from μ_+, for which they are valid, see Proposition 4.10 and 4.11. Firstly automorphism invariance follows, since if $T : \Omega \to \Omega$ is any automorphism then T^{-1} and $s_{H^+}^{-1}$ commute. Secondly, positive correlations follows since the mapping s_{H^+} is monotone. Thirdly, tail triviality follows because ν_+ has a smaller tail σ-field than μ_+: Let $X \in \Omega$ be a beach model realization and let $Y = s_{H^+}(X)$ be the sign configuration. If $A' \in \mathcal{T}' \subset \mathcal{F}_s$ is a tail event, then for each n we can determine whether A' occurs, i.e. $Y \in A'$, by observing the signs $Y(v_n), Y(v_{n+1}), \ldots$. Let $A = s_{H^+}^{-1}(A')$ and note that we, for every n, know if A occurs by just observing $X(v_n), X(v_{n+1}), \ldots$. Hence, A is a tail event for $(\Omega, \mathcal{F})$ and $\nu_+(A') = \mu_+(s_{H^+}^{-1}(A')) = \mu_+(A) =$ 0 or 1. □

5.3 Phase transition implies agreement percolation

For the Ising model, it is known that in the non-uniqueness regime where we have phase transition, the 'plus measure' of the Ising model exhibits agreement percolation for the ground state with all sites $\equiv +1$ [7].

How can one establish such a result? Coniglio et al. [7] developed a convenient criterion which can be used for general Markov random fields. A somewhat modified version is as follows.

Theorem 5.3 *Let (V, E) be a locally finite graph, μ a Markov random field on $\Omega = S^V$, and $H \subset \Omega$ a finite set of configurations. Suppose there exists a constant $c \in \mathbb{R}$ and a local function $f : \Omega \to \mathbb{R}$ depending only on the configuration in a connected set Δ, such that $\mu(f) > c$ but*

$$\mu(f|X \equiv \xi \text{ on } \partial\Gamma) \leq c \tag{10}$$

for all finite connected sets $\Gamma \supset \Delta$ and all $\xi \in \Omega$ with $s_H(\xi) \equiv 0$ on $\partial\Gamma$. Then $\mu(\Delta \overset{H}{\longleftrightarrow} \infty) > 0$, i.e. μ exhibits agreement percolation for H.

Proof. Suppose by contraposition that $\mu(\Delta \overset{H}{\longleftrightarrow} \infty) = 0$. For any $\varepsilon > 0$ we can then choose some finite $\Lambda \supset \Delta$ such that $\mu(\Delta \overset{H}{\longleftrightarrow} \Lambda^c) < \varepsilon$. For $\xi \notin \{\Delta \overset{H}{\longleftrightarrow} \Lambda^c\}$, there exists a connected set Γ such that $\Delta \subset \Gamma \subset \Lambda$ and $s_H(\xi) \equiv 0$ on $\partial\Gamma$; we simply let Γ be the union of Δ and all H-clusters meeting $\partial\Delta$. We let $\Gamma(\xi)$ be the largest such set. For $\xi \in \{\Delta \overset{H}{\longleftrightarrow} \Lambda^c\}$ we put $\Gamma(\xi) = \emptyset$. Then, for each finite connected set $\Gamma \neq \emptyset$, the event $\{\xi : \Gamma(\xi) = \Gamma\}$ depends only on the configuration in $\Lambda \setminus \Gamma$, whence by the Markov property $\mu(f|\Gamma(\cdot) = \Gamma)$ is an average of the conditional probabilities that appear in the assumption (10). From this we obtain

$$\mu(f) \leq c\mu(\Gamma(\cdot) \neq \emptyset) + \mu(|f|\mathbf{1}_{\{\Gamma(\cdot)=\emptyset\}}) < c + \varepsilon\|f\|.$$

Letting $\varepsilon \to 0$ we find that $\mu(f) \leq c$, contradicting our assumption. $\square$

Next we use the theorem above in the case of the beach model. One crucial set in Theorem 5.3 is $\{\xi : s_H(\xi) \equiv 0 \text{ on } \partial\Gamma \text{ for all finite } \Gamma \supset \Delta\}$. With $H = H^+$ this set corresponds to configurations with just -1 and -2 outside Δ.

Theorem 5.4 *If we have a phase transition for the beach model on G with origin o, then the 'plus sites' percolate in the 'plus measure':*

$$\mu_- \neq \mu_+ \implies \mu_+(o \overset{+}{\longleftrightarrow} \infty) > 0.$$

Proof. Assuming $\mu_- \neq \mu_+$ we have from Proposition 4.12 that $\mu_+(X(o) = +) > 1/2$. So, apply Theorem 5.3 with $\mu = \mu_+$, $H = H^+$, $c = 1/2$, $f = \mathbf{1}_{\{X(o)=+\}}$ and $\Delta = \{o\}$. Now check the condition (10). From Lemma 4.8 regarding beach model measures with given boundary we get

$$\mu_+(X(o) = +|X \equiv \xi \text{ on } \partial\Gamma) \leq \mu_+(X(o) = +|X \equiv -1 \text{ on } \partial\Gamma) \leq \frac{1}{2},$$

where the last inequality comes from the following calculation.

$$\mu_+(X(o) = +|X \equiv -1 \text{ on } \partial\Gamma) = 1 - \mu_+(X(o) = -|X \equiv -1 \text{ on } \partial\Gamma) =$$
$$1 - \mu_+(X(o) = +|X \equiv +1 \text{ on } \partial\Gamma) \leq 1 - \mu_+(X(o) = +|X \equiv -1 \text{ on } \partial\Gamma).$$

Here we have used the $\pm$-symmetry and again Lemma 4.8. The requirements of Theorem 5.3 are thus fulfilled and we conclude that $\mu_+(o \overset{+}{\longleftrightarrow} \infty) > 0$. □

The intuition behind the theorem is clear: If the plus sites do not percolate in the plus measure, there should a.s. be a contour of minuses surrounding the origin. But then, the origin itself could not possibly have a bias towards positive sign due to the Markov property.

5.4 Does agreement percolation imply phase transition?

A natural question is now if the converse of Theorem 5.4 holds. It turns out the answer depends on the graph. Taking the example $G = (\mathbb{Z}^d, \sim)$ we will show that in $d = 2$ dimensions the converse of Theorem 5.4 holds whereas for $d = 3$ dimensions it does not. We begin with the former statement.

5.4.1 The converse is true

Theorem 5.5 *For the beach model on $(\mathbb{Z}^2, \sim)$ phase transition is equivalent to agreement percolation:*

$$\mu_- \neq \mu_+ \iff \mu_+(o \overset{+}{\longleftrightarrow} \infty) > 0. \tag{11}$$

To prove this theorem we need some classical results on the number of infinite clusters for percolation models. First a definition.

Definition 5.6 A probability measure μ on $\{0,1\}^V$, with V a countable set, is said to have *finite energy* if, for every finite region $\Lambda \subset V$,

$$\mu(X \equiv \eta \text{ on } \Lambda | X \equiv \xi \text{ off } \Lambda) > 0$$

for all $\eta \in \{0,1\}^\Lambda$ and μ-a.e. $\xi \in \{0,1\}^{\Lambda^c}$.

The beach model lacks the finite energy property as -1 cannot sit next to $+2$ for example. However, looking at the signs only will give a model with finite energy.

Theorem 5.7 (The Burton-Keane uniqueness theorem) *Let μ be a probability measure on $\{0,1\}^{\mathbb{Z}^d}$ which is translation invariant and has finite energy. Then, μ-a.s., there exists at most one infinite open cluster.*

See [3] for a proof.

So, we can have at most one open cluster, and for obvious reasons, at most one closed cluster. Can they coexist? On $\mathbb{Z}^2$ (which is planar) it turns out they cannot. The following theorem is quoted from [9]. Their proof is based on a geometrical argument of Yu Zhang and of course Theorem 5.7.

Theorem 5.8 *Let μ be an automorphism invariant probability measure on $\{0,1\}^{\mathbb{Z}^2}$ with finite energy, positive correlation and trivial tail. Then*

$$\mu(\exists \text{ infinite open cluster}, \exists \text{ infinite closed cluster}) = 0.$$

Proof of Theorem 5.5. We only need to show the $\Leftarrow$-direction. Assume, to get a contradiction that $\mu_- = \mu_+$ and $\mu_+(o \stackrel{+}{\longleftrightarrow} \infty) > 0$. Focusing just on signs we then have $\nu_- = \nu_+$ and $\nu_+(o \longleftrightarrow \infty) > 0$. Consider the event that there exists an infinite open cluster. This is a tail event and from Proposition 5.2 ν_+ has trivial tail, and therefore this event must have ν_+-probability 1. ν_+ coincides with ν_- and by symmetry, an infinite closed cluster exists ν_+-a.s. This contradicts Theorem 5.8; the assumptions of this theorem are satisfied by Proposition 5.2. $\square$

5.4.2 The converse is false

The equivalence of non-uniqueness and percolation just observed on $\mathbb{Z}^2$ cannot be expected to hold for non-planar graphs. Consider, for example, the beach model on $\mathbb{Z}^3$. For $M = 1$ uniqueness certainly holds, and plus-percolation is here equivalent to Bernoulli percolation on $\mathbb{Z}^3$ with parameter $1/2$. But a result of [6] states that $p_c(\mathbb{Z}^3) < 1/2$. The plus spins thus percolate for $M = 1$. In view of the following theorem, this is still the case for sufficiently small $M > 1$, so that plus-percolation does occur in a non-trivial part of the uniqueness region.

Theorem 5.9 *There is some $M \in (1, M_c(\mathbb{Z}^3))$ such that the plus measure μ_+ for the beach model on $G = (\mathbb{Z}^3, \sim)$ exhibits agreement percolation for H^+,*

$$\exists M : 1 < M < M_c(\mathbb{Z}^3) \text{ and } \mu_+(o \stackrel{+}{\longleftrightarrow} \infty) > 0.$$

Proof. Note first from (7) that $M_c(\mathbb{Z}^3) > 1$. We have $\mu_+(o \stackrel{+}{\longleftrightarrow} \infty) = \nu_+(o \longleftrightarrow \infty)$, and since $\{o \longleftrightarrow \infty\}$ is an increasing event on Ω_s we are done if we can find $M < M_c(\mathbb{Z}^3)$ and $p > p_c(\mathbb{Z}^3)$ such that

$$\psi_p \preceq_{\mathcal{D}} \nu_+, \tag{12}$$

because for such a p we see from (9) that $\psi_p(o \longleftrightarrow \infty) > 0$. To establish (12) apply Holley's Theorem 2.4 to the projections of ψ_p and ν_+ on $\{0,1\}^{\Lambda_n}$, to get stochastic domination between the projected measures. The full stochastic domination (12) follows easily. Let Λ^* be short-hand for $\Lambda \setminus \{o\}$. We need to show that

$$\psi_p(X(o) = 1 | X(\Lambda_n^*) = \xi) \leq \nu_+(Y(o) = 1 | Y(\Lambda_n^*) = \eta) \tag{13}$$

for all $\xi, \eta \in \{0,1\}^{\Lambda_n^*}$ for which $\xi \preceq \eta$. The left-hand side equals p, of course. For the right-hand side, let X be a random field following μ_+. Then

$$\begin{aligned} \nu_+(Y(o) = 1 | Y(\Lambda_n^*) = \eta) &= \mu_+(X(o) = + | s_{H^+}(X(\Lambda_n^*)) = \eta) \\ &\geq \mu_+(X(o) = 1 | s_{H^+}(X(\Lambda_n^*)) = \eta). \end{aligned} \tag{14}$$

We name some relevant events: Let $A = \{X(o) = 1\}$, $B = \{s_{H^+}(X(\Lambda_n^*)) = \eta\}$ and $C = \{|X(\partial\{o\})| \equiv 1\}$. In the beach model every site has, given the configuration everywhere else, probability at least $1/M$ to take value in $\{-1, 1\}$. Therefore

$$\mu_+(C|B) \geq 1 - 2d(1 - \frac{1}{M}),$$

because the origin has $2d$ neighbours in $\mathbb{Z}^d$. Continuing from (14) we get

$$\begin{aligned} \mu_+(A|B) &\geq \mu_+(A \cap C|B) = \mu_+(A|C,B) \cdot \mu_+(C|B) \\ &\geq \frac{1}{M+1} \cdot (1 - 2d(1 - \frac{1}{M})), \end{aligned} \tag{15}$$

and combining (14) and (15) yields

$$\nu_+(Y(o) = 1|Y(\Lambda_n^*) = \eta) \geq \frac{1}{M+1} \cdot (1 - 2d(1 - \frac{1}{M})). \tag{16}$$

The right-hand side of (16) approaches $1/2$ as $M \searrow 1$, and since $p_c(\mathbb{Z}^3) < 1/2$, we can find some $p > p_c(\mathbb{Z}^3)$ satisfying (13) for M small enough. Now letting $n \to \infty$ will give (12) ending the proof. $\square$

It can be remarked that $p_c(\mathbb{Z}^d)$ is decreasing in the dimension d, so Theorem 5.9 is easily extended to higher dimensions $d \geq 3$.

6 The multi-coloured beach model

As mentioned before, the beach model and the Ising model have many similar properties. For example, every result above regarding the beach model has its Ising model analogue and they are all well known. The Potts model is the extension of the Ising model where, instead of having only two spin states ($-$ and $+$), there are q different spin states $(1, 2, \ldots, q)$, where $q \in \{2, 3, \ldots\}$. The Potts model with $q = 2$ corresponds to the Ising model. In [5] Burt and Steif introduced a corresponding generalization of the beach model. Let us look at it here, in the set-up with the reduced state space.

Let $G = (V, E) \in \mathcal{G}$ be some graph. Mark each vertex $x \in V$ with $\sigma_x = (c_x, j_x)$ from the state space $S = \{1, 2, \ldots, q\} \times \{1, 2\}$. The c_x will sometimes be referred to as the *colour* of the vertex x and the j_x as its *intensity*. A typical configuration $\sigma \in S^V = \Omega$ is a colouring of the vertices with different intensities. A configuration $\sigma = (c, j) \in \Omega$ is said to be a *BM-feasible* configuration if for $x, y \in V$,

$$x \sim y \Rightarrow \{c_x = c_y\} \vee \{j_x = j_y = 1\}.$$

Hence, in a BM-feasible configuration two neighbouring sites may have different colour only if they both have the lower intensity 1.

Let as before $\mathcal{F} = \sigma(\text{cylinder sets of } \Omega)$.

Definition 6.1 A probability measure μ on $(\Omega, \mathcal{F})$ is said to be a *Gibbs measure for the multi-coloured beach model on G with parameters $q \in \{2, 3, \ldots\}$ and* $M > 1$ if for all finite $\Lambda \subset V$, all $\sigma \in S^V$ and μ-a.a. $\eta \in S^{\Lambda^c}$ we have

$$\mu(X(\Lambda) = \sigma | X(\Lambda^c) = \eta) = \frac{1}{Z}(M-1)^{n_2(\sigma)} \mathbf{1}_{\{(\sigma \vee \eta) \text{ BM-feasible}\}}. \qquad (17)$$

Here n_2 is the number of vertices with the intensity 2. We see that μ has the Markov random field property (1) for the same reason as for the beach model defined in Definition 4.7. Note also that $q = 2$ in the model above would give a model equivalent to the beach model defined in Section 4 with 'colours' $-$ and $+$. In order to prove the existence of such measures, we will need the beach-random-cluster model (Section 6.1), so we will postpone this matter for a moment. However, for finite graphs, there is no problem of existence:

For a finite graph $G = (V, E)$, let μ_q^M be the Gibbs measure for the q-coloured beach model with parameter M, i.e. μ_q^M is the measure on S^V which to each $\sigma \in S^V$ assigns probability

$$\mu_q^M(\sigma) = \frac{1}{Z}(M-1)^{n_2(\sigma)} \mathbf{1}_{\{\sigma \text{ is BM-feasible}\}},$$

where again Z is a normalizing constant.

6.1 The random-cluster representation

The Fortuin–Kasteleyn random-cluster model has turned out to be of great value in analyzing the phase transition behavior of Ising and Potts models. Here we will look at a variant of the random-cluster model for the beach model, introduced in [13]. We start by defining it for finite graphs.

Let the graph $G = (V, E)$ be finite. For a site configuration $\xi \in \{0,1\}^V$, define the bond configuration $\xi^\star \in \{0,1\}^E$ by letting

$$\xi^\star(e) = \begin{cases} 1 & \text{if at least one of } e\text{'s endpoints takes value 1 in } \xi \\ 0 & \text{otherwise,} \end{cases}$$

for each $e \in E$.

Definition 6.2 *The beach-random-cluster measure $\phi_{p,q}$ for G with parameters $p \in [0,1]$ and $q > 0$* is the probability measure on $\{0,1\}^V$ which to each $\xi \in \{0,1\}^V = \Omega_s$ assigns probability

$$\phi_{p,q}(\xi) = \frac{1}{Z}\left\{\prod_{v \in V} p^{\xi(v)}(1-p)^{1-\xi(v)}\right\} q^{k^\star(\xi)}$$

where $k^\star(\xi)$ is the number of connected components in $\xi^\star$ (including isolated vertices) and Z is a normalizing constant.

Note that taking $q = 1$ yields the Bernoulli percolation measure ψ_p defined at the beginning of Section 5. All other choices of q give rise to dependencies between vertices (as long as p is not 0 or 1).

The beach-random-cluster model and the (multi-coloured) beach model itself are closely related. The following result is the key to using the random-cluster model in analyzing the beach model.

Proposition 6.3 *Let the graph $G = (V, E)$ be finite, let $p = (M-1)/M$, $q > 1$ an integer and suppose we pick a random beach model configuration $X = (c, j) \in S^V$ as follows:*

1. *Pick a random vertex configuration $Y \in \{0,1\}^V$ according to the random-cluster measure $\phi_{p,q}$.*
2. *For each connected component C of $Y^\star \in \{0,1\}^E$, pick a colour at random (uniformly) from $\{1, \dots, q\}$, assign this colour to every vertex in C and do this independently for different connected components.*
3. *Assign intensities by $j_x = Y(x) + 1, \forall x \in V$.*

Then X is distributed according to the Gibbs measure μ_q^M.

Proof. The proof is just a matter of counting. Let σ be a beach model configuration that was obtained through steps 1–3 from the random-cluster configuration η. Note that σ will be BM-feasible and that η is uniquely determined by σ. Thus

$$\begin{aligned} \mathrm{P}(\sigma) &= \mathrm{P}(\sigma|\eta) \cdot \mathrm{P}(\eta) = \left(\frac{1}{q}\right)^{k^\star(\eta)} \cdot \frac{1}{Z} p^{n_1(\eta)}(1-p)^{n_0(\eta)} q^{k^\star(\eta)} \\ &= \frac{1}{Z}\left(\frac{M-1}{M}\right)^{n_1(\eta)}\left(\frac{1}{M}\right)^{n_0(\eta)} = \frac{1}{Z}\frac{(M-1)^{n_2(\sigma)}}{M^{|V|}} \propto (M-1)^{n_2(\sigma)}, \end{aligned}$$

where of course $n_1(\eta)$ and $n_0(\eta)$ represent the number of open and closed vertices in η. $\square$

As a warm-up for the phase transition considerations, we give the following result as a typical application of the random-cluster representation.

Corollary 6.4 *If we pick a random beach configuration $X = (c, j) \in S^V$ according to the Gibbs measure μ_q^M, then for $i \in \{1, \ldots, q\}$ and two vertices $x, y \in V$, the two events $\{c_x = i\}$ and $\{c_y = i\}$ are positively correlated, i.e.*

$$\mu_q^M(c_x = i, c_y = i) \geq \mu_q^M(c_x = i) \cdot \mu_q^M(c_y = i).$$

Proof. The measure μ_q^M is invariant under permutations of the colour set $\{1, \ldots, q\}$, so that

$$\mu_q^M(c_x = i) = \mu_q^M(c_y = i) = \frac{1}{q}.$$

We therefore need to show that

$$\mu_q^M(c_x = i, c_y = i) \geq \frac{1}{q^2}.$$

We may now think of X as being obtained as in Proposition 6.3 by first picking a configuration $Y \in \{0,1\}^V$ according to the random-cluster measure $\phi_{p,q}$ and then assigning i.i.d. uniform colours to the connected components. Given Y, the conditional probability that $c_x = c_y = i$ is $1/q$ if x and y are in the same connected component of $Y^\star$, and $1/q^2$ if they are in different connected components. Hence, for some $\alpha \in [0,1]$,

$$\mu_q^M(c_x = i, c_y = i) = \alpha \frac{1}{q} + (1-\alpha)\frac{1}{q^2} \geq \frac{1}{q^2}.$$

$\square$

6.1.1 Infinite-volume limits

Definition 6.2 cannot be applied immediately to infinite graphs, but there are natural generalizations, so called thermodynamic limits. In this subsection we will exploit some stochastic monotonicity properties of random-cluster distributions on finite subgraphs of $\mathbb{Z}^d$. This will give us the existence of certain limiting random-cluster distributions on $\mathbb{Z}^d$, and also the existence of Gibbs measures for the beach model.

The basic observation is stated in the lemma below which follows directly from definitions.

Lemma 6.5 *Consider the beach-random-cluster model with parameters p and q on a finite graph $G = (V, E)$. For any vertex $x \in V$, and any configuration $\eta \in \{0,1\}^{V \setminus \{x\}}$, we have*

$$\phi_{p,q}(x \text{ is open} \mid \eta) = \frac{p\, q^{1-k^\star(x,\eta)}}{p\, q^{1-k^\star(x,\eta)} + 1 - p}, \tag{18}$$

where $k^\star(x,\eta)$ is the number of connected components containing either x or some neighbour of x, in the edge configuration $\eta^\star$ corresponding to η.

For $q \geq 1$, Lemma 6.5 means in particular that the conditional probability in (18) is increasing both in η and p. This allow us to use Holley's Theorem and the FKG inequality to prove the following result.

Corollary 6.6 *For a finite subgraph G of $\mathbb{Z}^d$ with the beach-random-cluster measure $\phi_{p,q}$ with $p \in [0,1]$ and $q \geq 1$, we have*

(a) *$\phi_{p,q}$ is monotone, and therefore has positive correlations,*

(b) *$\phi_{p,q} \preceq_{\mathcal{D}} \psi_p$,*

(c) *$\phi_{p,q} \succeq_{\mathcal{D}} \psi_{\widehat{p}}$,*

where

$$\widehat{p} = \frac{p}{p + (1-p)q^{2d-1}}.$$

Furthermore, for $0 \leq p_1 \leq p_2 \leq 1$ and $q \geq 1$, we have

(d) *$\phi_{p_1,q} \preceq_{\mathcal{D}} \phi_{p_2,q}$.*

Proof. The monotonicity in (a) is just the observation that the conditional probability in (18) is increasing in p and η. Positive correlations follows from Theorem 2.6. Next, note that (18) and $1 \leq k^\star(x,\eta) \leq 2d$ imply that

$$\frac{p}{p + (1-p)q^{2d-1}} \leq \phi_{p,q}(x \text{ is open} \,|\, \eta) \leq p \tag{19}$$

for all η as in Lemma 6.5. Theorem 2.4 in conjunction with the second (resp. first) inequality in (19) implies (b) (resp. (c)). Finally, (d) is just another application of Theorem 2.4. $\square$

Consider now the integer lattice $G = (V,E) = (\mathbb{Z}^d, \sim)$ (for simplicity) and let $(\Lambda_n)_{n=1}^\infty$ be some exhaustion of it, such that Λ_n^c is connected for all n. We associate with each finite region $\Lambda_n \subset \mathbb{Z}^d$ a specific random-cluster distribution with a certain boundary condition. Let $\phi_{p,q,n}$ be the probability measure on $\{0,1\}^V$ for which each $\xi \in \{0,1\}^V$ is assigned probability

$$\phi_{p,q,n}(\xi) = \frac{1}{Z}\left\{\prod_{v\in\Lambda_n} p^{\xi(v)}(1-p)^{1-\xi(v)}\right\} q^{k^\star(\xi,\Lambda_n)}\, \mathbf{1}_{\{\xi\equiv 1 \text{ off } \Lambda_n\}}, \tag{20}$$

where $k^\star(\xi,\Lambda_n)$ is the number of connected components in $\xi^\star$ (including isolated vertices) that do not intersect Λ_n^c. For the boundary condition $\{\xi \equiv 1 \text{ off } \Lambda_n\}$, all sites of Λ^c may be thought of as being firmly wired together by $\xi^\star$, why this is called the *wired boundary condition.* There is also the *free boundary condition*: $\{\xi \equiv 0 \text{ off } \Lambda_n\}$, which we will not make use of here.

Consider the random-cluster measures $\phi_{p,q,n}$ and $\phi_{p,q,n+1}$. Since $\Lambda_n \subset \Lambda_{n+1}$, we can obtain $\phi_{p,q,n}$ from $\phi_{p,q,n+1}$ by conditioning on the event $\{\xi \equiv 1 \text{ on } \Lambda_{n+1} \setminus \Lambda_n\}$ which is an increasing event. Hence, if $q \geq 1$ then Corollary 6.6(a) implies that

$$\phi_{p,q,n} \succeq_{\mathcal{D}} \phi_{p,q,n+1} \text{ for } n = 1,2,\ldots,$$

by complete analogy with (4). Moreover, we obtain the following counterpart of Proposition 4.9 on the existence of infinite-volume limits.

Lemma 6.7 *For $p \in [0,1]$ and $q \geq 1$, the limiting measure*

$$\phi_{p,q} = \lim_{n\to\infty} \phi_{p,q,n},$$

exists and is translation invariant.

This convergence result has consequences for the convergence of Gibbs distributions for the multi-coloured beach model, as we will show next. For $q \in \{2, 3, \ldots\}$ and $i \in \{1, \ldots, q\}$ and for any member of the exhaustion Λ_n let $\mu^i_{q,n}$ denote the beach model measure on Ω with boundary condition $\eta \equiv (i, 2)$ on Λ^c_n, as in (17). The dependence of M for these measures is here suppressed. In the same way as for finite graphs, a beach model configuration σ following $\mu^i_{q,n}$ can be obtained starting with a random-cluster configuration chosen randomly according to $\phi_{p,q,n}$. The following is a simple modification of Proposition 6.3 and the proof is analogous.

Proposition 6.8 *Let the graph $G = (V, E) = (\mathbb{Z}^d, \sim)$, $p = (M-1)/M$ and suppose we pick a random beach model configuration $X = (c, j) \in S^V$ as follows:*

1. *Pick a random vertex configuration $Y \in \{0,1\}^V$ according to the random-cluster measure $\phi_{p,q,n}$.*
2. *For each* finite *connected component C of $Y^\star$, pick a colour at random (uniformly) from $\{1, \ldots, q\}$, assign this colour to every vertex in C and do this independently for different connected components.*
3. *The remaining vertices are assigned the colour i.*
4. *Assign intensities by $j_x = Y(x) + 1, \forall x \in V$.*

Then X is distributed according to the Gibbs measure $\mu^i_{q,n}$.

Corollary 6.9 *With $p = (M-1)/M$, we have*

$$\mu^i_{q,n}(c_o = i) = \frac{1}{q}\left\{1 + (q-1)\phi_{p,q,n}(o \stackrel{\star}{\longleftrightarrow} \Lambda^c_n)\right\}.$$

Here $\stackrel{\star}{\longleftrightarrow}$ refers to connectivity in the edge configuration $Y^\star$.

Proof. Using the recipe in Proposition 6.8 we see that the origin will get colour i for sure if it is connected to the boundary and with probability $1/q$ otherwise. We have

$$\begin{aligned}
\mu^i_{q,n}(c_o = i) &= \phi_{n,p,q}(o \stackrel{\star}{\longleftrightarrow} \Lambda^c_n) + \left\{1 - \phi_{n,p,q}(o \stackrel{\star}{\longleftrightarrow} \Lambda^c)\right\}\frac{1}{q} \\
&= \frac{1}{q}\left\{1 + (q-1)\phi_{n,p,q}(o \stackrel{\star}{\longleftrightarrow} \Lambda^c_n)\right\}.
\end{aligned}$$

□

Proposition 6.8 leads us to the following result extending Proposition 4.9 to the multi-coloured beach model.

Proposition 6.10 *For any $i \in \{1, \ldots, q\}$, the limiting probability measure*

$$\mu^i_q = \lim_{n\to\infty} \mu^i_{q,n}$$

on $S^{\mathbb{Z}^d}$ exists and is a translation invariant Gibbs measure for the multi-coloured beach model on $\mathbb{Z}^d$ with parameters q and M. The limit is independent of the choice of exhaustion.

Proof. As before, a weak limit of finite-volume Gibbs measures is a Gibbs measure whenever it exists. We thus need to show that $\mu^i_{q,n}(f)$ converges as $n \to \infty$, for any local function f.

Fix an f as above, and let $\Delta \subset \mathbb{Z}^d$ be the region on which f depends. Take n large enough so that $\Delta \subset \Lambda_n$. As shown above, we may think of a S^V-valued random element X with distribution $\mu^i_{q,n}$ as arising by first picking a random-cluster configuration $Y \in \{0,1\}^V$ according to $\phi_{p,q,n}$ (with $p = (M-1)/M$) and then assigning random colours to the connected components, forcing colour i to the (unique) infinite cluster. For $x, y \in \Delta$, we write $\{x \leftrightarrow y\}$ for the event that x and y are in the same connected component in $Y^\star$, and $\{x \leftrightarrow \infty\}$ for the event that x is in an infinite cluster. Clearly, the conditional distribution of f given Y depends only on the indicator functions $(\mathbf{1}_{\{x\leftrightarrow y\}})_{x,y\in\Delta}$, $(\mathbf{1}_{\{x\leftrightarrow\infty\}})_{x\in\Delta}$ and $(Y(x))_{x\in\Delta}$, since the conditional distribution of X on Δ is uniform over all elements of S^Δ such that the recipe in Proposition 6.8 is followed. Hence, the desired convergence of $\mu^i_{q,n}$ follows if we can show that the joint distribution of $(\mathbf{1}_{\{x\leftrightarrow y\}})_{x,y\in\Delta}$, $(\mathbf{1}_{\{x\leftrightarrow\infty\}})_{x\in\Delta}$ and $(Y(x))_{x\in\Delta}$ converges as $n \to \infty$. This, however, follows from Lemma 6.7 and an inclusion-exclusion argument upon noting that $(\mathbf{1}_{\{x\leftrightarrow y\}})_{x,y\in\Delta}$, $(\mathbf{1}_{\{x\leftrightarrow\infty\}})_{x\in\Delta}$ and $(Y(x))_{x\in\Delta}$ are increasing functions. □

6.2 Phase transition

In this subsection we will see that the beach model exhibits phase transition for certain choices of the parameters.

Consider the multi-coloured beach model on $\mathbb{Z}^d$, $d \geq 2$. All the arguments to be used here, except those showing that the critical value $M_c(q)$ is strictly between 1 and ∞, go through on arbitrary infinite graphs; we stick to the $\mathbb{Z}^d$ case for simplicity of notation. We consider the limiting Gibbs measures μ^i_q obtained in Proposition 6.10. These play a role similar to that of the 'plus' and 'minus' measures μ_+ and μ_- in Section 4. The difference is that those measures represented extreme Gibbs measures in the sense of stochastic ordering, whereas the measures μ^i_q cannot be compared in the same way. We have already seen effects of this in the proof of Proposition 6.10, where the use of the random-cluster representation was essential.

We first state an analog of Corollary 6.6, applicable to the random-cluster measures $\phi_{p,q,n}$ on $\mathbb{Z}^d$ with wired boundaries. We write $\psi_{p,n}$ for the probability measure on $\{0,1\}^{\mathbb{Z}^d}$ which to a configuration ξ assigns probability

$$\psi_{p,n}(\xi) = \prod_{x\in\Lambda_n} p^{\xi(x)}(1-p)^{1-\xi(x)}\mathbf{1}_{\{\xi\equiv 1 \text{off } \Lambda_n\}}.$$

Lemma 6.11 *For the beach-random-cluster measure $\phi_{p,q,n}$ with $p \in [0,1]$, $q \geq 1$ and $n \in \mathbb{N}$, we have*

(a) *$\phi_{p,q,n}$ is monotone,*

(b) *$\phi_{p,q,n} \preceq_{\mathcal{D}} \psi_{p,n}$,*

(c) *$\phi_{p,q,n} \succeq_{\mathcal{D}} \psi_{\widehat{p},n}$,* *where*

$$\widehat{p} = \frac{p}{p + (1-p)q^{2d-1}}.$$

Furthermore, for $0 \le p_1 \le p_2 \le 1$ *and* $q \ge 1$*, we have*

(d) $\phi_{p_1,q,n} \preceq_{\mathcal{D}} \phi_{p_2,q,n}$.

Proof. Single-site conditional probabilities for $x \in \Lambda_n$ under $\phi_{p,q,n}$ are the same as in the finite setting (18). For any $\eta \in \{0,1\}^{\Lambda_n \setminus \{x\}}$,

$$\phi_{p,q,n}(x \text{ is open} \,|\, \eta) = \frac{p\, q^{1-k^\star(x,\eta)}}{p\, q^{1-k^\star(x,\eta)} + 1 - p}.$$

We are thus back in the situation of Corollary 6.6, and the proof of that corollary also applies here. □

The following result is a variant of Proposition 4.12 for the multi-coloured beach model. It also gives a characterization of phase transition in terms of percolation in the random-cluster model.

Theorem 6.12 *Let* $M > 1$ *and* $p = (M-1)/M$*. For any* $x \in \mathbb{Z}^d$ *and any* $i \in \{1, \ldots, q\}$*, the following statements are equivalent.*

(i) $\mu_q^1 = \mu_q^2 = \ldots = \mu_q^q$,

(ii) $\mu_q^i(c_x = i) = 1/q$,

(iii) $\phi_{p,q}(x \stackrel{\star}{\longleftrightarrow} \infty) = 0$.

Proof. (i)⇒(ii): By symmetry $\mu_q^i(c_x = i) = \mu_q^j(c_x = j)$ for any $i, j \in \{1, \ldots, q\}$. From (i) we must then have $\mu_q^i(c_x = i) = 1/q$, and (ii) is established.

(ii)⇒(iii): For the given vertex x, take an exhaustion $(\Lambda_n)_{n=1}^\infty$ where $x \in \Lambda_1$. Using Corollary 6.9 we have

$$\mu_q^i(c_x = i) = \lim_{n\to\infty} \mu_{q,n}^i(c_x = i) = \frac{1}{q} + \frac{q-1}{q} \lim_{n\to\infty} \phi_{p,q,n}(x \stackrel{\star}{\longleftrightarrow} \Lambda_n^c).$$

But, from Lemma 6.7 we get

$$\lim_{n\to\infty} \phi_{p,q,n}(x \stackrel{\star}{\longleftrightarrow} \Lambda_n^c) = \lim_{n\to\infty} \phi_{p,q,n}(x \stackrel{\star}{\longleftrightarrow} \infty) = \phi_{p,q}(x \stackrel{\star}{\longleftrightarrow} \infty), \tag{21}$$

thus proving the implication.

(iii)⇒(i): We will show that $\mu_q^i = \mu_q^j$ for any $i, j \in \{1, \ldots, q\}$, by showing that $\mu_q^i(f) = \mu_q^j(f)$ for all local functions $f : \Omega \to \mathbb{R}$. Fix such a function f and some $\varepsilon > 0$. For a given exhaustion $(\Lambda_n)_{n=1}^\infty$ we then can find some n_0 such that f only depends on the configuration on Λ_{n_0} and such that

$$\left|\mu_q^i(f) - \mu_{q,n}^i(f)\right| < \varepsilon \text{ for all } n > n_0. \tag{22}$$

In view of (iii) and (21) we can also find an $m > n_0$ such that $\phi_{p,q,m}(x \stackrel{\star}{\longleftrightarrow} \Lambda_m^c) < \varepsilon/|\Lambda_{n_0}|$ for all $x \in \Lambda_{n_0}$, and thus

$$\phi_{p,q,m}(\Lambda_{n_0} \stackrel{\star}{\longleftrightarrow} \Lambda_m^c) < \varepsilon. \tag{23}$$

Here $\{\Lambda_{n_0} \stackrel{\star}{\longleftrightarrow} \Lambda_m^c\}$ is the event that there exists an open path from Λ_{n_0} to Λ_m^c. Denote this event C.

Now, let us make use of Proposition 6.8. Take $Y \in \{0,1\}^{\mathbb{Z}^d}$ according to $\phi_{p,q,m}$ and let it generate two beach model configurations $X^i, X^j \in \Omega$ coupled in the following way: In step 2 of Proposition 6.8, let the finite components get the same colour in both configurations X^i and X^j. The infinite components get colours i and j, respectively. Then X^i will follow $\mu^i_{q,m}$ and X^j will follow $\mu^j_{q,m}$. Because f is local it has finite range, so without loss of generality we can assume $\|f\| \leq 1$. Using (23) we then get

$$\begin{aligned} \left|\mu^i_{q,m}(f) - \mu^j_{q,m}(f)\right| &= \left|\mu^i_{q,m}(f\mathbf{1}_C) + \mu^i_{q,m}(f\mathbf{1}_{C^c}) - \mu^j_{q,m}(f\mathbf{1}_C) - \mu^j_{q,m}(f\mathbf{1}_{C^c})\right| \\ &= \left|\mu^i_{q,m}(f\mathbf{1}_C) - \mu^j_{q,m}(f\mathbf{1}_C)\right| \\ &\leq 2\,\phi_{p,q,m}(C) \\ &< 2\,\varepsilon. \end{aligned} \tag{24}$$

Combining (22) and (24) gives

$$\left|\mu^i_q(f) - \mu^j_q(f)\right| < 4\,\varepsilon,$$

and, since ε was arbitrary, we know that μ^i_q and μ^j_q are identical on cylinder sets, and hence identical. □

Note that the missing link (ii)⇒(iii) in Proposition 4.12 now follows as a special case of the last theorem. Just take $q = 2$ and use the implication (ii)⇒(i) of Theorem 6.12. Needless to say, none of the results leading to Theorem 6.12 rely on Proposition 4.12.

In Theorem 6.12, the statement (i) does not exclude the possibility of the existence of more than one Gibbs measure, i.e. phase transition. But in fact, we have the following result.

Theorem 6.13 *Each one of the statements* (i)-(iii) *of Theorem 6.12 are equivalent to*

(iv) *There is a unique Gibbs measure for the q-coloured beach model on $\mathbb{Z}^d$ with parameter M.*

We omit the proof, which is a tedious but straightforward adaptation of the corresponding result in [9] for the Potts model. The idea for proving (iii)⇒(iv) is that absence of percolation in the random-cluster model implies that every region is cut off from infinity by a set of closed edges. Thus, independently of what happens macroscopically, the local spins feel as if they are in a system with free boundary condition. This makes a phase transition impossible.

Now we have the tools to prove that the phenomena of phase transition for the multi-coloured beach model is increasing in M. We will see the usefulness of the percolation criterion (iii) above, and get a generalization of Theorem 4.13.

Theorem 6.14 *There exists a critical value $M_c(q) \in [1, \infty]$ such that for $M < M_c(q)$ we have that the multi-coloured beach model on $\mathbb{Z}^d$ with parameters q and M has a unique Gibbs measure whereas for $M > M_c(q)$ there are multiple Gibbs measures.*

Proof. Assume $M_1 < M_2$. We are to show that if there are multiple Gibbs measures for the beach model with parameters M_1 and q, then so is the case also for the model with the parameters M_2 and q.

Let $p_1 = (M_1 - 1)/M_1$ and $p_2 = (M_2 - 1)/M_2$. Then $p_1 < p_2$ and from Lemma 6.11(d) we have $\phi_{p_1,q,n} \preceq_{\mathcal{D}} \phi_{p_2,q,n}$ for $n = 1, 2, \ldots$. Going to the limit, we get $\phi_{p_1,q} \preceq_{\mathcal{D}} \phi_{p_2,q}$.

Assume there are multiple Gibbs measures when $M = M_1$. Then, by using the equivalence of (iii) and (iv) in Theorem 6.13, we have that $\phi_{p_1,q}(x \overset{\star}{\longleftrightarrow} \infty) > 0$. The event $\{x \overset{\star}{\longleftrightarrow} \infty\}$ is increasing and therefore $\phi_{p_2,q}(x \overset{\star}{\longleftrightarrow} \infty) > 0$ from the above. Thus, using Theorem 6.13 again, we have multiple Gibbs measures also for $M = M_2$. $\square$

The reader who was disappointed for not getting the proof of Theorem 6.13 can instead define $M_c(q)$ as the point $\sup_M\{\text{condition (i) holds}\}$.

The percolation criterion in Theorem 6.12 (iii) raises the question of when a non-trivial percolation threshold exists for the beach-random-cluster model. For $\mathbb{Z}^d$ this is answered below.

Proposition 6.15 *For the beach-random-cluster model on $\mathbb{Z}^d$, $d \geq 2$, and any fixed $q \geq 1$, there exists a percolation threshold $p_c(q) \in (0, 1)$ (depending on d) such that*

$$\phi_{p,q}(o \overset{\star}{\longleftrightarrow} \infty) \begin{cases} = 0 & \text{for } p < p_c(q), \\ > 0 & \text{for } p > p_c(q). \end{cases}$$

Proof. The statement of the proposition consists of the following three parts:

(i) $\phi_{p,q}(o \overset{\star}{\longleftrightarrow} \infty) = 0$ for p sufficiently small,

(ii) $\phi_{p,q}(o \overset{\star}{\longleftrightarrow} \infty) > 0$ for p sufficiently close to 1, and

(iii) $\phi_{p,q}(o \overset{\star}{\longleftrightarrow} \infty)$ is increasing in p.

We first prove (i). Take some exhaustion $(\Lambda_n)_{n=1}^{\infty}$ of $\mathbb{Z}^d$ and suppose $p < p_c(\mathbb{Z}^d)$, the critical value for Bernoulli site percolation on $\mathbb{Z}^d$ defined in (9). For $\varepsilon > 0$, we can then pick n large enough so that

$$\psi_p(o \overset{\star}{\longleftrightarrow} \Lambda_n^c) \leq \varepsilon.$$

Let $m > n$, so that $\Lambda_m \supset \Lambda_n$. By Lemma 6.11(b), we have

$$\begin{aligned} \phi_{p,q,m}(o \overset{\star}{\longleftrightarrow} \infty) &\leq \phi_{p,q,m}(o \overset{\star}{\longleftrightarrow} \Lambda_n^c) \\ &\leq \psi_{p,m}(o \overset{\star}{\longleftrightarrow} \Lambda_n^c) = \psi_p(o \overset{\star}{\longleftrightarrow} \Lambda_n^c) \\ &\leq \varepsilon. \end{aligned}$$

Since ε was arbitrary, we find by first letting $m \to \infty$,

$$\phi_{p,q}(o \overset{\star}{\longleftrightarrow} \infty) = 0, \quad \text{for } p < p_c(\mathbb{Z}^d),$$

proving (i).

Next, (ii) can be established by a similar argument: Let p be such that $\widehat{p} = p/[p + (1-p)q^{2d-1}] > p_c(\mathbb{Z}^d)$. Lemma 6.11(c) then shows that $\phi_{p,q,n} \succeq_{\mathcal{D}} \psi_{\widehat{p},n} \succeq_{\mathcal{D}} \psi_{\widehat{p}}$ for every n, so that

$$\phi_{p,q}(o \overset{\star}{\longleftrightarrow} \infty) = \lim_{n\to\infty} \phi_{p,q,n}(o \overset{\star}{\longleftrightarrow} \infty) \geq \psi_{\widehat{p}}(o \overset{\star}{\longleftrightarrow} \infty) > 0,$$

proving (ii).

To check (iii), use Lemma 6.11(d). This proves (iii) and thereby the proposition. □

In [5] it was shown that for $M > 2e \cdot q(7q^2)^d$, there are exactly q different Gibbs measures for the q-coloured beach model on $\mathbb{Z}^d$. Here we know from Theorem 6.14 that the model has a critical M-value and the next theorem says it satisfies $1 < M_c(q) < \infty$.

Theorem 6.16 *For $G = (\mathbb{Z}^d, \sim)$, $d \geq 2$, and $q \in \{2, 3, \dots\}$, the q-coloured beach model has a critical value $M_c(q) \in (1, \infty)$.*

Proof. Let $p_c(q) \in (0,1)$ be the critical p-value for the beach-random-cluster model on $\mathbb{Z}^d$, and let

$$M_0 = \frac{1}{1 - p_c(q)}$$

Now take $M < M_0$ and note that $p = (M-1)/M$ will satisfy $p < p_c(q)$. Thus $\phi_{p,q}(0 \overset{\star}{\longleftrightarrow} \infty) = 0$ and from Theorem 6.12 there is a unique Gibbs measure for the corresponding beach model. On the other hand, if $M > M_0$, then $p > p_c(q)$ and again by Theorem 6.12, we have a phase transition. Thereby the critical $M_c(q)$ and M_0 must coincide. Finally, $p_c(q) \in (0,1)$ implies that $M_c(q) \in (1, \infty)$. □

7 The beach model on a regular tree

In this concluding section we will make use of the previous results in a special case: the beach model on a regular tree. It turns out that the simple structure of this graph makes it possible to compute the Gibbs measure marginal at the root. As we know, this is closely connected to the question of uniqueness of Gibbs measures. Here this question can be transferred to the question of the number of solutions to a certain fixed point problem. To this question, numerical methods give the answer, or at least an educated guess.

Let $\mathbb{T}_k$ denote the infinite rooted tree with k children in each generation. All vertices have $k+1$ neighbours, except for the root, which has k neighbours. Let the n:th generation be the vertices at distance n from the root. Also, let Γ_n be the finite subtree of $\mathbb{T}_k$, which includes all vertices up to, and including, generation n. The root will be denoted 0.

We will start by looking more closely at the beach random-cluster model on $V = \{\text{the vertex set of } \mathbb{T}_k\}$. As in Section 6, we define random-cluster measures on an increasing family of subgraphs. For example, $(\Gamma_n)_{n=1}^{\infty}$ constitutes a good exhaustion for this purpose. Let $\phi_{p,q,n}$ be the probability measure on $\{0,1\}^V$ for which each $\xi \in \{0,1\}^V$ is assigned probability

$$\phi_{p,q,n}(\xi) = \frac{1}{Z}\left\{\prod_{v\in\Gamma_n} p^{\xi(v)}(1-p)^{1-\xi(v)}\right\} q^{k^\star(\xi,\Gamma_n)}\,\mathbf{1}_{\{\xi\equiv 1 \text{ off } \Gamma_n\}}, \tag{25}$$

where $k^\star(\xi,\Gamma_n)$ is the number of connected components in $\xi^\star$ (including isolated vertices) that do not intersect Γ_n^c. This is similar to (20), and the difference is that here Γ_n^c is not connected. We will however still refer to this as the *wired boundary condition*. For $q \geq 1$ the same stochastic domination relations hold as in Section 6 and a limiting probability measure is guaranteed:

Lemma 7.1 *For $p \in [0,1]$ and $q \geq 1$, the limiting probability measure*

$$\phi_{p,q} = \lim_{n\to\infty} \phi_{p,q,n},$$

exists on $\{0,1\}^V$.

Next, we go on to the beach model on $\mathbb{T}_k$. As before the state space is $S = \{1,\ldots,q\}\times\{1,2\}$, and the configuration space is now $\Omega = S^V$. Let $\mu^i_{q,k,n}$ be the beach model measure on Ω which is $\equiv (i,2)$ on Γ_n^c and else follows the right-hand side of (17). A configuration following $\mu^i_{q,k,n}$ can be obtained exactly as in Proposition 6.8, starting with a random-cluster configuration from $\phi_{p,q,n}$. Note that step 3 in Proposition 6.8 makes sure that $\mu^i_{q,k,n}$ delivers proper beach model configurations on Γ_n with an all i-coloured boundary. Thus, the wired random-cluster configurations for a regular tree can be thought of as being connected “at infinity”. The corresponding approach to defining the Fortuin–Kasteleyn model on trees was introduced in [12]. As before, the dependence of the parameter M is suppressed when denoting the measure $\mu^i_{q,k,n}$.

7.1 The magnetization at the root

For the Ising model, the mean $\mu(X(o))$ is sometimes referred to as the magnetization at the origin. This value, of course, is directly connected to the probability of having a plus at the origin:

$$\mu(X(o)) = \mu(X(o) = +1) - \mu(X(o) = -1) = 2\mu(X(o) = +1) - 1.$$

Note that there is no magnetization if, and only if, $\mu(X(o) = +1) = 1/2$. The corresponding probability for the beach model will be of interest here.

Proposition 6.10 can be adapted to the $\mathbb{T}_k$-case, by use of Lemma 7.1. Therefore, as $n \to \infty$, the limit $\mu^i_{q,k,n} \Rightarrow \mu^i_{q,k}$ exists as a weak limit and is a beach model Gibbs measure for $\mathbb{T}_k$. The interesting question is whether there are any other Gibbs measures? We answer this question by using Theorem 6.12 and Theorem 6.13, which go through on any infinite graph. The relevant statements are extracted in the following remark.

Remark 7.2 *Let $M > 1$ and let $q, k \in \{2, 3, \ldots\}$, with $i \in \{1, \ldots, q\}$. Then the following two statements are equivalent.*

(i) $\mu^i_{q,k}(c_0 = i) = 1/q$.

(ii) *There is a unique Gibbs measure on $\mathbb{T}_k$ with parameters q and M.*

In view of Remark 7.2 we would like to compute the probability

$$\theta_{q,k}(M) = \mu^i_{q,k}(c_0 = i),$$

i.e. the probability that the root has the same colour as the boundary. We start by dealing with the same probability for the finite trees Γ_n. Let

$$\theta_{q,k,n}(M) = \mu^i_{q,k,n}(c_0 = i). \tag{26}$$

Then

$$\theta_{q,k}(M) = \lim_{n\to\infty} \theta_{q,k,n}(M),$$

since the sequence of probabilities are decreasing and thus converging.

Let us look at the beach model configurations on Γ_n with boundary condition of colour i outside Γ_n. Think of M as an integer again, as in Section 4. Recall that the beach model measure is the the uniform measure over of all BM-feasible configurations. Hence, computing the probability of finding the colour i at the root is only a matter of comparing the number of i-rooted configurations with the number of possible ones.

To shorten notations, represent the boundary condition colour i with $+$. All the other colours are by symmetry exchangeable, and we represent one of them with $-$. Intensities are represented with 1 and 2 as before, but think of the 2 as being $(M-1)$-multiple. Let $E_n \subset S^{\Gamma_n}$ be the set of BM-feasible configurations on Γ_n that also meet the boundary condition of $+2$ on $\partial\Gamma_n$. We can partition E_n into $2q$ sets, depending on the configuration at the root. Let

$$A^n_{+2} = |\{X : X \in E_n \text{ and } X(0) = +2\}|$$

and define A^n_{+1}, A^n_{-1} and A^n_{-2} analogously. Also, let $A^n = |E_n| = A^n_{+2} + A^n_{+1} + (q-1)(A^n_{-1} + A^n_{-2})$. With this notation we get

$$\theta_{q,k,n}(M) = \frac{A^n_{+1} + A^n_{+2}}{A^n}.$$

A nice property of the regular tree is that the numbers $A^n_{\cdot\cdot}$ can be obtained recursively. When determining the number of configurations of size Γ_{n+1} with a given root configuration, we look at the k vertices in generation 1 and their subtrees of Γ_n-size, recognizing a smaller problem. With some thought and the BM-feasibility in mind we then get the following relations:

$$\begin{cases} A^0_{+2} &= M-1, \\ A^0_{+1} &= 1, \\ A^0_{-1} &= 0, \\ A^0_{-2} &= 0, \end{cases}$$

and, for $n \geq 0$,

$$\begin{cases} A^{n+1}_{+2} &= \left(A^n_{+2} + A^n_{+1}\right)^k (M-1), \\ A^{n+1}_{+1} &= \left(A^n_{+2} + A^n_{+1} + (q-1)A^n_{-1}\right)^k, \\ A^{n+1}_{-1} &= \left(A^n_{+1} + (q-1)A^n_{-1} + A^n_{-2}\right)^k, \\ A^{n+1}_{-2} &= \left(A^n_{-1} + A^n_{-2}\right)^k (M-1). \end{cases} \tag{27}$$

Inspired by the recursion (27), let the mapping $T_1 : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

$$T_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (x_1+x_2)^k(M-1) \\ (x_1+x_2+(q-1)x_3)^k \\ (x_2+(q-1)x_3+x_4)^k \\ (x_3+x_4)^k(M-1) \end{bmatrix}.$$

Upon noting that $T_1([1\ 0\ 0\ 0]) = [M-1\ 1\ 0\ 0]$, we can rewrite (27) as

$$[A^n_{+2}\ A^n_{+1}\ A^n_{-1}\ A^n_{-2}] = T_1^{n+1}([1\ 0\ 0\ 0]), \quad n \geq 0.$$

To get the desired probabilities we need to normalize by dividing with A^n. Therefore, let $T_2 : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

$$T_2(x) = \frac{x}{x_1 + x_2 + (q-1)(x_3+x_4)}$$

for those $x = [x_1\ x_2\ x_3\ x_4]$ where the denominator does not vanish. Now the distribution for the root configuration of Γ_n is given by $(T_2 \circ T_1^{n+1})(x_0)$. It is not hard to see that the result would not be altered if we do some in between rescaling, i.e. $T_2 \circ T_1^{n+1} = (T_2 \circ T_1)^{n+1}$. Therefore, define the mapping $T : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$T = T_2 \circ T_1, \tag{28}$$

and we get the following lemma.

Lemma 7.3 *Let $\theta_{q,k,n}(M)$ be as defined in (26), and T as in (28). Then*

$$\theta_{q,k,n}(M) = [1\ 1\ 0\ 0] \bullet T^{n+1}([1\ 0\ 0\ 0]),$$

where $\bullet$ denotes scalar product.

7.2 A fixed point problem

Consider the following fixed point problem.

(P) Let $M > 1$, $q > 1$ and $k > 1$ be real numbers and let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be as in (28). Solve the equation

$$T(x) = x.$$

Remark 7.4 *If x is a solution to* **(P)***, then $T_2(x) = T_2(T_2T_1\, x) = T_2^2\, T_1\, x = T_2T_1\, x = x$. Hence, the solution x satisfies $x_1 + x_2 + (q-1)(x_3 + x_4) = 1$.*

We will see that the number of solutions to **(P)** is connected to the number of Gibbs measures for the beach model on $\mathbb{T}_k$. First two lemmas.

Lemma 7.5 *Let k be an integer. The vector $\widehat{x}$ is a solution to* **(P)***, where*

$$\widehat{x} = \lim_{n\to\infty} T^n([1\ 0\ 0\ 0]).$$

Proof. Let $x_n = T^{n+1}([1\ 0\ 0\ 0])$. Then x_n is the marginal probability distribution of $\mu^i_{q,k,n}$ at the root. As $n \to \infty$ the sequence $(x_n)_{n=0}^{\infty}$ will converge to the root marginal for $\mu^i_{q,k}$. Since T is a continuous mapping, the limit $\widehat{x} = \lim_{n\to\infty} x_n$ solves the problem **(P)**. □

Lemma 7.6 *The vector $\widetilde{x}$ is a solution to* **(P)***, where*

$$\widetilde{x} = [\, a \quad 1/q - a \quad 1/q - a \quad a\,],$$

and $a \in (0, 1/q)$ solves the equation

$$a = \frac{M-1}{q\,\{M - 1 + (q + aq - aq^2)^k\}}. \tag{29}$$

Proof. With $\widetilde{x}$ as above we get

$$T(\widetilde{x}) = \frac{1}{q\,\{(1/q)^k(M-1) + (1 + a - aq)^k\}} \begin{bmatrix} (1/q)^k(M-1) \\ (1+a-aq)^k \\ (1+a-aq)^k \\ (1/q)^k(M-1) \end{bmatrix}.$$

Apparently, $T(\widetilde{x})$ is symmetric in the same way as $\widetilde{x}$, and the sum of its first two components equals $1/q$. Thus, $\widetilde{x}$ solves **(P)** if a is chosen so that $\widetilde{x}_1 = [T(\widetilde{x})]_1$, i.e. a satisfies (29). In fact, there is (at least) one solution to (29) for which $0 < a < 1/q$: Let

$$g(a) = a - \frac{M-1}{q\,\{M - 1 + (q + aq - aq^2)^k\}},$$

and we can check that $g(0) = -(M-1)/(q(M-1) + q^{k+1}) < 0$, $g(1/q) = 1/(qM) > 0$ and g is continuous. □

We call a solution to **(P)** symmetric if it is on the form $[a \ \ 1/q - a \ \ 1/q - a \ \ a]$, and the solution $\widetilde{x}$ of Lemma 7.6 is therefore symmetric. In the same way as $\widehat{x}$ corresponds to the root marginal for $\mu^i_{q,k}$, the symmetric $\widetilde{x}$ corresponds to the root marginal for a Gibbs measure on $\mathbb{T}_k$ with free boundary.

Proposition 7.7 *Let $M > 1$ be real, and let $q > 1$, $k > 1$ be integers. Then the solution $\widehat{x}$ in Lemma 7.5 is symmetric, if, and only if, there is a unique Gibbs measure on $\mathbb{T}_k$ with parameters M and q.*

Proof. If $\widehat{x}$ is symmetric, we have

$$\begin{aligned}\theta_{q,k}(M) &= [1\ 1\ 0\ 0] \bullet \widehat{x} = a + (1/q - a) \\ &= 1/q,\end{aligned}$$

and Remark 7.2 then gives the uniqueness.

If there is a unique Gibbs measure, then Remark 7.2 implies that $\widehat{x}_1 + \widehat{x}_2 = 1/q$. From Remark 7.4 we then see that $\widehat{x}$ must be on the form $\widehat{x} = [a \;\; 1/q - a \;\; 1/q - b \;\; b]$ for some $b \in [0, 1/q]$. Finally,

$$\frac{a}{b} = \frac{\widehat{x}_1}{\widehat{x}_4} = \frac{[T(\widehat{x})]_1}{[T(\widehat{x})]_4} = \frac{(M-1)/q^k}{(M-1)/q^k} = 1,$$

proving that $\widehat{x}$ is symmetric. □

Corollary 7.8 *Let $M > 1$ be real, and let $q > 1$, $k > 1$ be integers. Then*

(**P**) *has exactly one solution $\Rightarrow$ There is a unique Gibbs measure on $\mathbb{T}_k$.*

Proof. If (**P**) has exactly one solution, then the solutions $\widehat{x}$ and $\widetilde{x}$ of Lemma 7.5 and Lemma 7.6 must coincide. Hence, $\widehat{x}$ is symmetric and Proposition 7.7 gives the result. □

Although q, k and sometimes even M have been thought of as integers, there is no need for such a restriction on the parameters for (**P**). The problem (**P**) can be posed for all real $M \geq 1, q \geq 1, k \geq 1$, and could thus be seen as an extension of the beach model on a regular tree – at least regarding the question of phase transition.

It is of interest to compare these trees with the homogeneous ones. They have one additional branch at the root, making the degree of every vertex in the tree equal. A homogeneous tree is therefore transitive. Let $\mathbb{T}_k'$ be homogeneous tree where every vertex has $k+1$ neighbours. To compute the magnetization at the origin for $\mathbb{T}_k'$, we do the same procedure as for $\mathbb{T}_k$ with the only difference that in the last iteration we substitute $k+1$ for k. It is thus clear that we get positive magnetization for $\mathbb{T}_k'$ if, and only if, we get it for $\mathbb{T}_k$. The two trees therefore have the same critical value.

7.3 Numerics

Proposition 7.7 suggests a way of determining whether the parameter triplet (M, q, k) allows for phase transition or not:

1. Choose $\varepsilon > 0$ and $\delta > 0$ small with $\varepsilon \ll \delta$, and let $x_0 = [1\ 0\ 0\ 0]$.
2. Compute $x_{n+1} = T(x_n)$ and iterate until $\|x_{n+1} - x_n\|_\infty < \varepsilon$.

3. Guess that (M, q, k) allows for only one Gibbs measure if

$$\left|[1\ 1\ 0\ 0] \bullet x_n - \frac{1}{q}\right| < \delta,$$

otherwise not.

For computation in MATLAB, in which calculations are done with an accuracy of order 10^{-16}, the choices for ε and δ have been $\varepsilon = 10^{-13}$ and $\delta = 10^{-7}$.

$q\backslash k$	1.5	2	3	4
1.5	1.961	1.595	1.399	1.350
2	2.493	2.113	2.030	1.952
3	3.544	3.130	2.698	2.381
4	4.487	3.868	3.109	2.648

Table 1: Critical values $M_c(q, k)$.

We can now estimate the critical value $M_c(q, k)$ for some different q and k, see Table 1. A plot of the M_c surface for small q and k is shown in Figure 1.

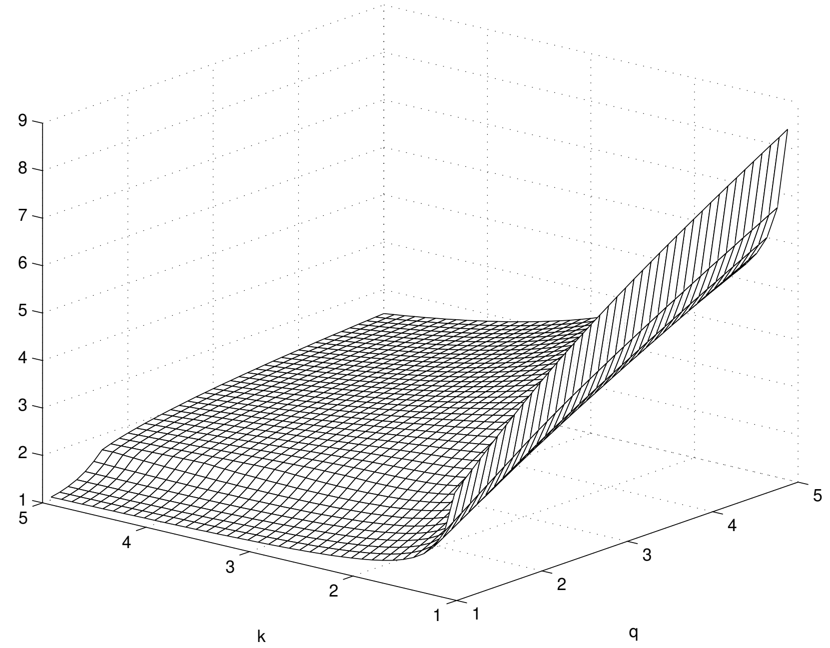

Figure 1: Plot of $M_c(q, k)$.

We can also plot the "magnetization probability" $\theta_{q,k}(M) \approx \theta_{q,k,n}(M)$ for some n suggested by step 2 above. We see three examples in Figures 2, 3 and 4. In Figure 2 and 3, it is likely that $\theta_{q,k}(M)$ is discontinuous, but in the third example, where $q = k = 2$, $\theta_{q,k}(M)$ looks continuous. In physics language, we say there is a first order phase transition in the former cases, whereas for $q = k = 2$ we only have a second order phase transition.

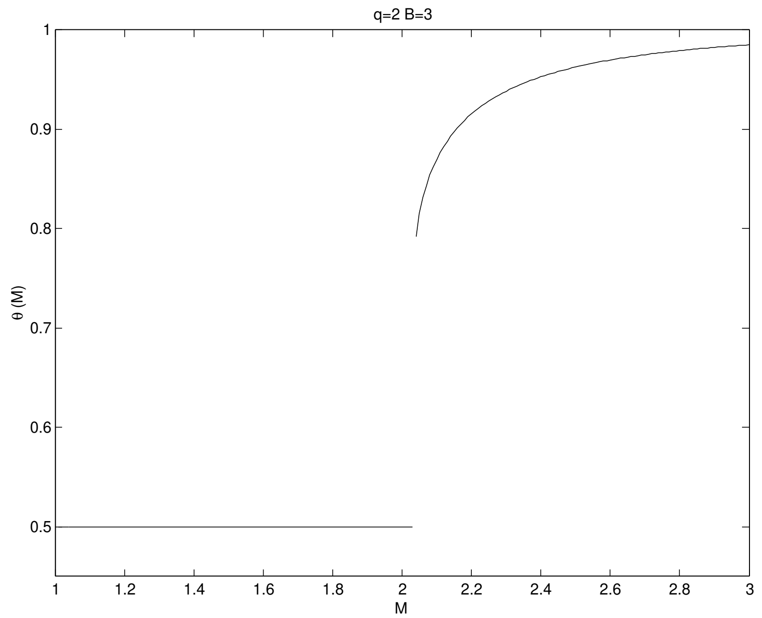

Figure 2: Plot of $\theta_{q,k}(M)$ in the case $q = 2$, $k = 3$.

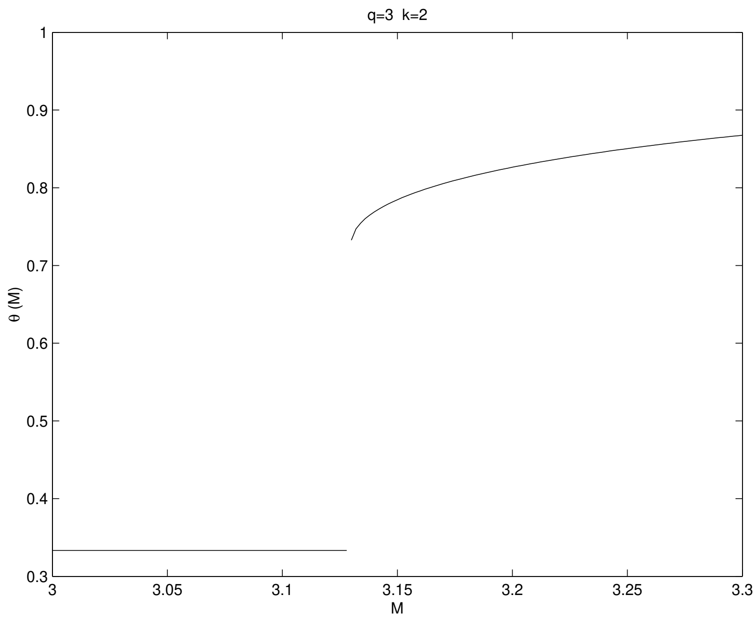

Figure 3: Plot of $\theta_{q,k}(M)$ in the case $q = 3$, $k = 2$.

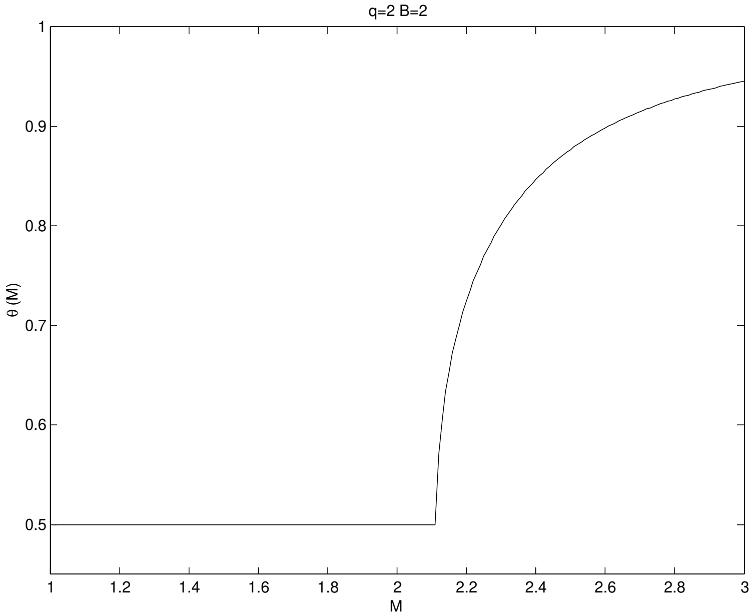

Figure 4: Plot of $\theta_{q,k}(M)$ in the case $q = 2$, $k = 2$.

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