COVER TIMES OF A SIMPLE RANDOM WALK WITH EXPONENTIAL HOLDING TIMES

Anna Carlsund

Abstract

A simple random walk on the integers with independent, exponentially distributed holding times is stopped when it has covered a range of length m. When and where does it stop? In what way does it differ from a simple random walk with constant holding times? What kind of limit process do we get if m is large and the process is appropriately scaled?

SIMPLE RANDOM WALK; COVER TIMES; RANGE; BROWNIAN MOTION; LAPLACE TRANSFORMS.

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Contents

1 Introduction

Consider a simple random walk, $\{S_n\}_{n\geq 0}$, starting in 0 with probability p of jumping one step up and probability q of jumping one step down. We define the cover time of an interval of length m as

$$N \stackrel{\text{def}}{=} \min\{n : \max_{0\leq i\leq n} S_i - \min_{0\leq i\leq n} S_i = m\}. \tag{1}$$

N is the minimal number of steps it takes for the random walk to visit m different states, except the initial state 0.

In Figure 1(a) a realization is given of a simple symmetric ($p = q = \frac{1}{2}$) random walk $\{S_n\}_{n\geq 0}$ which is stopped when it has visited $m = 6$ different states except 0. Here N is 22. The random walk ends in state 4, so the endpoint is $S_N = 4$ in this case.

Questions arise as what distribution does N have, what distribution does the endpoint S_N have and if time and space are scaled in an appropriate way what happens for large m? Such questions are investigated in Chong, Cowan and Holst [2], where also further references on the subject can be found.

Suppose now that the holding times are exponentially distributed with intensity $\lambda+\mu$ instead of being constant. The Markov chain jumps one step up with intensity λ and one step down with intensity μ. Since the holding times are exponentially distributed, the probability of jumping one step up is $p = \frac{\lambda}{\lambda+\mu}$ and one step down is $q = \frac{\mu}{\lambda+\mu}$. We get a continuous time Markov chain, a special type of a birth and death process. Let $\{S_t\}_{t\geq 0}$ denote this random process and define the cover time T of m states as

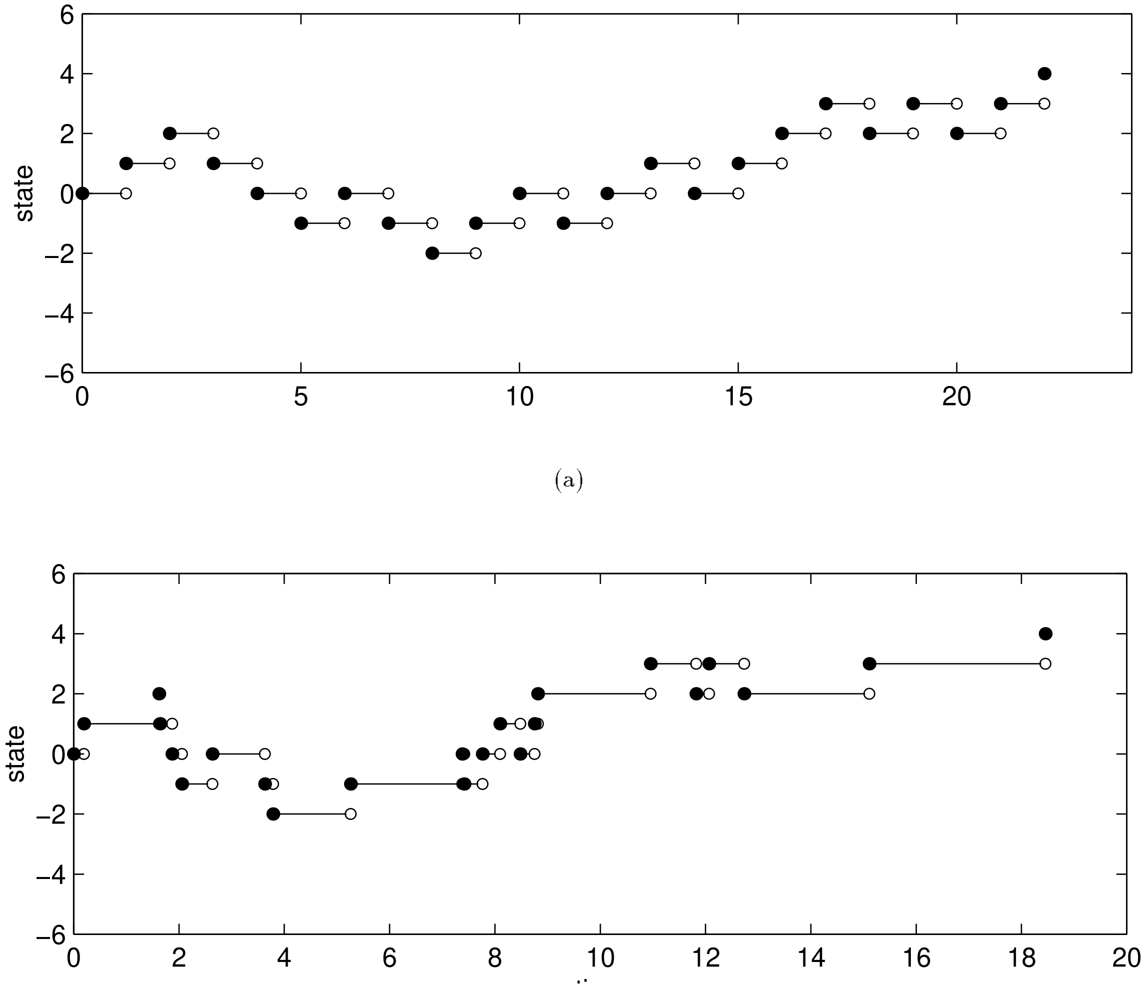
$$T \stackrel{\text{def}}{=} \inf\{t : \max_{0\leq s\leq t} S_s - \min_{0\leq s\leq t} S_s = m\}. \tag{2}$$

In Figure 1(b) a realization is given of such a continuous time Markov chain with $\lambda = \mu = \frac{1}{2}$ which is stopped when it has visited 6 different states. Here $T = 18.45$. The random walk ends in state 4, so the endpoint $S_T = 4$.

The same questions as above we ask for the continuous time Markov chain. In Section 2 we discuss the cover time of this kind of continuous time Markov chain and in Section 3 the range covered a predetermined time t is discussed. In Section 4 this process is scaled and a limit process is obtained. Section 5 contains some concluding remarks.

2 The continuous cover time T

Let the holding times, $\{H_i\}_{i\geq 1}$, be independent and exponentially distributed with intensity $\lambda + \mu$ and the only possibilities of a jump is one step up or one step down



(a)

(b)

Figure 1: In (a) a realization of a simple symmetric random walk is given. In (b) a realization of a continuous time Markov chain with exponentially distributed holding times with mean 1 and the same realization of the jump chain as in (a) is given.

(as before). Then $\{S_t\}_{t\geq 0}$ is a continuous time Markov chain. The only difference between the simple random walk and the continuous time Markov chain $\{S_t\}_{t\geq 0}$ just described is that the holding times are not constant any longer but exponentially distributed with intensity $\lambda+\mu$. From now and on we denote T by T_N and we have

$$T_N = \sum_{i=1}^{N} H_i, \tag{3}$$

where N is defined in (1). The process, $\{S_t\}t \geq 0$ is a birth and death process which also takes negative values.

Theorem 2.1 *The density function for the cover time T_N is*

$$f_{T_N}(t) = e^{-t(\lambda+\mu)}\sqrt{4pq}(\lambda+\mu)\Bigg[\sum_{k=1}^{m-1}\theta(k,m)\left(e^{\tau(k,m,t)} - \sum_{n=0}^{m-2}\frac{\tau(k,m,t)^n}{n!}\right)$$
$$-\sum_{k=1}^{m}\theta(k,m+1)\left(e^{\tau(k,m+1,t)} - \sum_{n=0}^{m-2}\frac{\tau(k,m+1,t)^n}{n!}\right)\Bigg] \tag{4}$$

for $t \geq 0$ where

$$\theta(k,m) = \frac{1}{m(pq)^{(m-1)/2}}(-1)^k\frac{(p^{m/2}-(-1)^k q^{m/2})^2}{1-2\sqrt{pq}\cos\frac{\pi k}{m}}\sin^2\frac{\pi k}{m} \tag{5}$$

and

$$\tau(k,m,t) = \sqrt{4pq}t(\lambda+\mu)\cos(\frac{\pi k}{m}).$$

Proof. The cover time T_N is a random sum of independent exponentially distributed variables. A sum of independent, identically exponentially distributed random variables is gamma distributed. Thus as N and $\{H_i\}_{i\geq 1}$ are all independent, the sum (3) conditioned on N is gamma distributed. We have

$$F_{T_N}(t) = P\left(\sum_{i=1}^{N} H_i \leq t\right) = \sum_{n=m}^{\infty} P\left(\sum_{i=1}^{N} H_i \leq t | N=n\right)P(N=n)$$

$$= \sum_{n=m}^{\infty} P\left(\sum_{i=1}^{n} H_i \leq t\right)P(N=n) = \sum_{n=m}^{\infty}\int_0^t \frac{1}{\Gamma(n)}x^{n-1}(\lambda+\mu)^n e^{-x(\lambda+\mu)}dx P(N=n)$$

$$= \int_0^t \sum_{n=m}^{\infty}\frac{1}{\Gamma(n)}x^{n-1}(\lambda+\mu)^n e^{-x(\lambda+\mu)}P(N=n)dx.$$

Hence the density is

$$f_{T_N}(t) = \sum_{n=m}^{\infty} \frac{1}{\Gamma(n)} t^{n-1} (\lambda+\mu)^n e^{-t(\lambda+\mu)} P(N=n). \tag{6}$$

It is proved in Chong, Cowan and Holst [2] Theorem 3 that

$$P(N=n) = (4pq)^{n/2} \Big(\sum_{k=1}^{m-1} \theta(k,m) \cos^{n-1} \frac{\pi k}{m} - \sum_{k=1}^{m} \theta(k,m+1) \cos^{n-1} \frac{\pi k}{m+1} \Big) \tag{7}$$

for $n \geq 3$ where $\theta(k,m)$ is defined in (5) and $P(N=2) = p^2+q^2$ for $m=2$. Using this explicit expression for $P(N=n)$ in (6) and summing over n we get

$$\begin{aligned} f_{T_N}(t) = e^{-t(\lambda+\mu)} \sqrt{4pq} (\lambda+\mu) \Big[& \sum_{k=1}^{m-1} \theta(k,m) \Big(e^{\tau(k,m,t)} - \sum_{n=0}^{m-2} \frac{\tau(k,m,t)^n}{n!} \Big) \\ & - \sum_{k=1}^{m} \theta(k,m+1) \Big(e^{\tau(k,m+1,t)} - \sum_{n=0}^{m-2} \frac{\tau(k,m+1,t)^n}{n!} \Big) \Big]. \end{aligned}$$

□

If the expected step length is 1, the simple random walk, S_n, and the continuous time Markov chain, S_t can be compared. In Figure 2 are some examples given. The alternating probabilities for a random walk depending on if N is odd or even is due to combinatorics (see [2], page 6 footnote (i)). These irregularities are smoothed out in the density function of T_N.

The expected cover time is

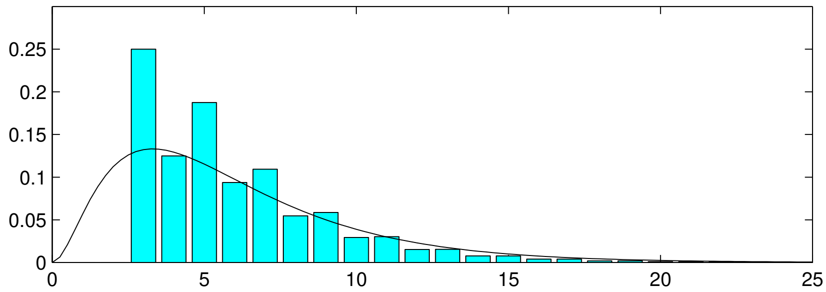
$$E[T_N] = E[\sum_{i=1}^{N} H_i] = E[E[\sum_{i=1}^{N} H_i | N]] = E[N]E[H_i] = E[N]\frac{1}{\lambda+\mu}.$$

For $\lambda+\mu=1$ the expected length of the steps are 1 and the expected cover time is the same for a simple random walk and the Markov process. $E[N]$ is derived in Chong, Cowan and Holst [2],

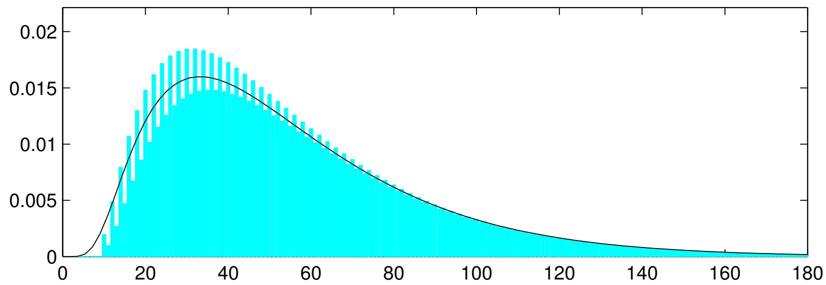
$$E[N] = \frac{m(p-q)(p^m-q^m)(p^{m+1}+q^{m+1}) - m^2(p-q)^2(pq)^m - pq(p^m-q^m)^m}{(p-q)^2(p^m-q^m)(p^{m+1}-q^{m+1})}$$

for $p \neq q$ and $E[N] = \frac{m(m+1)}{2}$ for $p=q=1/2$.

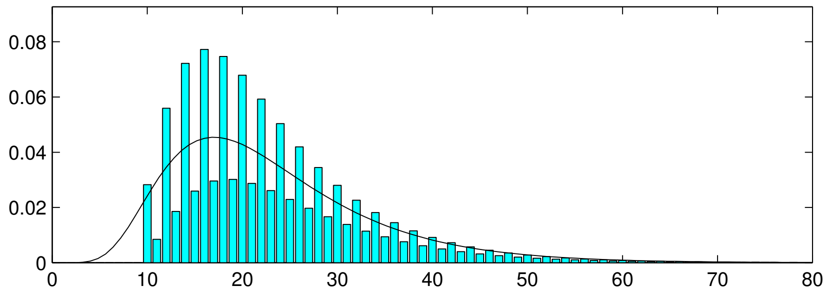
Since the jump chain and the holding times are independent the distribution of the endpoint, S_{T_N}, is the same as the distribution of S_N. Actually the distribution



(a) m=3, p=q=0.5



(b) m=10, p=q=0.5



(c) m=10, p=0.7, q=0.3

Figure 2: The bars are the probability function of N and the solid line is the density function of T_N. The expected holding time is $\lambda + \mu = 1$ and $p = \lambda$ and $q = \mu$. The tails are cut off.

of the endpoint does not depend on whether or not the holding times are constant as long as the holding times are independent of the jump chain. Thus from Chong, Cowan and Holst [2] we have

$$P(S_{T_N} = k) = p^{m+(k-|k|)/2} q^{m-(k+|k|)/2} \frac{(p^{|k|} - q^{|k|})(p-q)}{(p^m - q^m)(p^{m+1} - q^{m+1})}$$

for $p \neq q$ and $P(S_{T_N} = k) = |k|/(m(m+1))$ for $p = q = 1/2$.

3 The range covered at time t

Instead of considering the time it takes to visit a predetermined number of states as in Section 2, consider how many states that are covered at a predetermined time, that is the range R_t at time t

$$R_t \stackrel{\text{def}}{=} \max_{0 \le s \le t} S_s - \min_{0 \le s \le t} S_s.$$

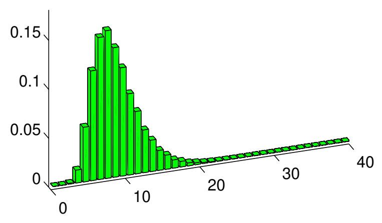
The probability function of R_t can be calculated in the same way as in Chong, Cowan and Holst [2] for a random walk, using that $\{R_t < m\} = \{T_N > t\}$ and the explicit formula for $P(T_N \le t)$. To avoid confusion, let N_m and T_{N_m} denote N and T_N respectively where the index m is (as before) the number of different states the chain has visited when stopped. We have

$$\begin{aligned} &P(R_t = m) = P(R_t < m+1) - P(R_t < m) = P(T_{N_{m+1}} > t) - P(T_{N_m} > t) \\ &= P(T_{N_m} \le t) - P(T_{N_{m+1}} \le t) = \int_0^t \frac{1}{\Gamma(m)} x^{m-1} (\lambda + \mu)^m e^{-x(\lambda+\mu)} dx P(N_m = m) \\ &\quad + \sum_{n=m+1}^{\infty} \int_0^t \frac{1}{\Gamma(n)} x^{n-1} (\lambda+\mu)^n e^{-x(\lambda+\mu)} dx (P(N_m = n) - P(N_{m+1} = n)). \quad (8) \end{aligned}$$

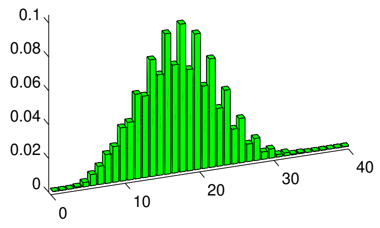
The formula for $P(R_t = m)$, (8), is complicated but useful for numerical calculations. In Figure 3 we see the same phenomena as in Figure 2, the probabilities of R_n are alternating and smoothed out for R_t, cf. [2] page 6 footnote (iv).

4 The limiting case of a Markov chain with exponential holding times

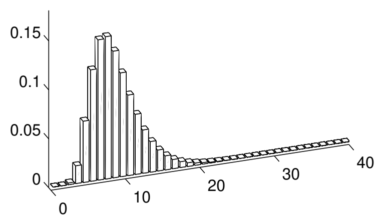
If the length of the steps and the time between the steps of the random walk are scaled in an appropriate way, then the random walk converges weakly to a Brownian motion when the number of steps tend to infinity. This enables us to derive some



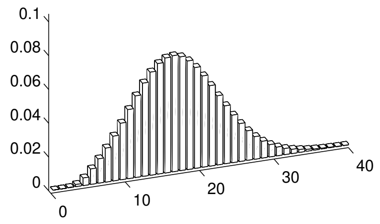
(a) $p = q = 0.5$



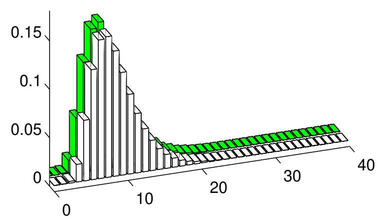
(b) $p = 0.7$



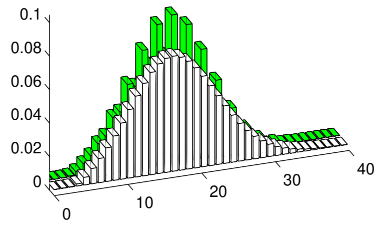
(c) $\lambda = \mu = 0.5$



(d) $\lambda = 0.7$



(e) $p = \lambda = 0.5$



(f) $p = \lambda = 0.7$

Figure 3: Examples of the probability distribution of R_t (light bars) and R_n (dark bars) with $t = 40$. The tails of the probability function of R_t are cut off at t=40 in (c), (d), (e) and (f).

properties about the limit of the random walk, such as the cover time for example. In Chong, Cowan and Holst [2] this is done by first deriving the probability generating function of $(M, N-M, S_N)$, where M is the number of steps it takes the random walk to reach the opposite endpoint of the coverage interval for the first time. If we apply this method to the continuous time Markov chain, the Laplace transform for $(T_M, T_N - T_M, S_{T_N})$ is needed, where T_N is used instead of T and T_M denotes the first time the opposite endpoint of the coverage interval is reached. For an illustration see Figure 1(a) where $M = 8$ and Figure 1(b) where $T_M = 3.86$.

4.1 The Laplace transform for $(T_M, T_N - T_M, S_{T_N})$

With the notations above

$$T_M = \sum_{i=1}^{M} H_i$$

where H_i are exponentially distributed with mean $l = \frac{1}{\lambda+\mu}$. The Laplace transform for $(T_M, T_N - T_M, S_{T_N})$ is easily derived using the probability generating function for $(M, N-M, S_N)$ calculated in [2] Theorem 2,

$$E[x^M y^{N-M} z^{S_N}]$$

$$= \left(\frac{\mathrm{sh}((m+1)\eta(x))\mathrm{ch}(m\zeta(z)) - \mathrm{sh}(m\eta(x))\mathrm{ch}((m+1)\zeta(z)) - \mathrm{sh}(\eta(x))}{(\mathrm{ch}(\eta(x)) - \mathrm{ch}(\zeta(z)))\mathrm{sh}(m\eta(x))\mathrm{sh}((m+1)\eta(y))} \right) \mathrm{sh}(\eta(y)), \tag{9}$$

where $\mathrm{sh}(\cdot) \overset{\mathrm{def}}{=} \sinh(\cdot)$, $\mathrm{ch}(\cdot) \overset{\mathrm{def}}{=} \cosh(\cdot)$, $\eta(x) \overset{\mathrm{def}}{=} \operatorname{arccosh} \frac{1}{x\sqrt{4pq}}$ and $\zeta(z) \overset{\mathrm{def}}{=} \ln\sqrt{\frac{p}{q}} + \ln z$.

Theorem 4.1 *Let $\{S_t\}_{t\geq 0}$ be a continuous time Markov chain which only can jump one step up or one step down and with exponentially distributed holding times with mean l. Set $\zeta = \ln\sqrt{\frac{p}{q}}$ and $\chi(sl) = \operatorname{arccosh}\left(\frac{1+sl}{\sqrt{4pq}}\right)$. Then*

$$E[e^{-sT_M - w(T_N - T_M) - zS_{T_N}}]$$
$$= \left(\frac{\mathrm{sh}((m+1)\chi(sl))\,\mathrm{ch}(m(\zeta-z)) - \mathrm{sh}(m\chi(sl))\,\mathrm{ch}((m+1)(\zeta-z)) - \mathrm{sh}(\chi(sl))}{(\mathrm{ch}(\chi(sl)) - \mathrm{ch}(\zeta-z))\,\mathrm{sh}(m\chi(sl))\,\mathrm{sh}((m+1)\chi(wl))} \right)$$
$$\times\, \mathrm{sh}(\chi(wl)).$$

Proof.

$$E[e^{-sT_M-w(T_N-T_M)-zS_{T_N}}]=E[E[e^{-sT_M-w(T_N-T_M)-zS_{T_N}}|N,M,S_{T_N}]]$$

$$=E[e^{-zS_{T_N}}E[e^{-s\sum_{i=1}^{M}H_i-w\sum_{i=1}^{N-M}H_{M+i}}|N,M,S_{T_N}]].$$

Using that $\{H_i\}_{i\geq 1}$ are independent exponentially distributed random variables with mean l and independent of (M,N) the expression above becomes

$$E\left[e^{-zS_{T_N}}E\left[\left(\frac{1}{1+sl}\right)^M\left(\frac{1}{1+wl}\right)^{N-M}|N,M,S_{T_N}\right]\right]$$

$$=E\left[e^{-zS_{T_N}}\left(\frac{1}{1+sl}\right)^M\left(\frac{1}{1+wl}\right)^{N-M}\right].$$

As the endpoint does not depend on the holding times i.e. $S_{T_N}=S_N$ and using (9) the assertion follows. □

4.2 The limiting case of a Markov chain with exponential holding times

To get a limit case, the length of the steps and the time between the steps need to be appropriately scaled. Let the length of the steps be c and the mean time between the steps be l. Set

$$p=\frac{1}{2}\left(1+\frac{c\nu}{\sigma^2}\right)\qquad q=\frac{1}{2}\left(1-\frac{c\nu}{\sigma^2}\right)\qquad c^2=\sigma^2 l \tag{10}$$

for some ν and some $\sigma>0$. This gives the mean replacement $\frac{c}{l}(p-q)=\nu$ per unit of time with a variance $\frac{c^2}{l}=\sigma^2$. Letting the length of the steps tend to 0 and thus the intensity to infinity we get the limit of the continuous time Markov chain.

Theorem 4.2 *Let $\{S_t\}_{t\geq 0}$ be a Markov chain with exponentially distributed holding times with mean l and the only possibilities for a jump is one step up or one step down. Let the relations (10) hold then we have*

$$\lim_{c\to 0}E[e^{-sT_M-w(T_N-T_M)-zcS_{T_N}}]$$

$$=\frac{2\alpha(w)}{\mathrm{sh}(\alpha(s))\,\mathrm{sh}(\alpha(w))}\left(\frac{\mathrm{sh}^2(\frac{\alpha(s)+\varepsilon-rz}{2})}{\alpha(s)+\varepsilon-rz}+\frac{\mathrm{sh}^2(\frac{\alpha(s)-\varepsilon+rz}{2})}{\alpha(s)-\varepsilon+rz}\right)$$

where $\mathrm{sh}(\cdot)\stackrel{\mathrm{def}}{=}\sinh(\cdot)$, $\varepsilon\stackrel{\mathrm{def}}{=}\frac{r\nu}{\sigma^2}$ *and* $\alpha(x)\stackrel{\mathrm{def}}{=}\frac{r}{\sigma}\sqrt{2x+\frac{\nu^2}{\sigma^2}}$.

Proof. Using Theorem 4.1 we get

$$E[e^{-sT_M - w(T_N - T_M) - zcS_{T_N}}]$$
$$= \left(\frac{\mathrm{sh}((m+1)\chi(sl))\mathrm{ch}(m(\zeta - zc)) - \mathrm{sh}(m\chi(sl))\mathrm{ch}((m+1)(\zeta - zc)) - \mathrm{sh}(\chi(sl))}{(\mathrm{ch}(\chi(sl)) - \mathrm{ch}(\zeta - zc))\mathrm{sh}(m\chi(sl))\mathrm{sh}((m+1)\chi(wl))}\right)$$
$$\times\, \mathrm{sh}(\chi(wl)). \quad (11)$$

To see the limit behaviour of this Laplace transform when $c \to 0$, we first arrange (11) as (12) and we derive the limits in (13), (14), (15) and (16). The Laplace transform can be written

$$\Bigg(\frac{\mathrm{sh}\big(\frac{m}{2}(\chi(sl) + \zeta - cz)\big)\mathrm{sh}\big(\frac{m+1}{2}(\chi(sl) + \zeta - cz)\big)}{\mathrm{sh}\big(\frac{1}{2}(\chi(sl) + \zeta - cz)\big)\mathrm{sh}\big(m\chi(sl)\big)}$$
$$+ \frac{\mathrm{sh}\big(\frac{m}{2}(\chi(sl) - \zeta + cz)\big)\mathrm{sh}\big(\frac{m+1}{2}(\chi(sl) - \zeta + cz)\big)}{\mathrm{sh}\big(\frac{1}{2}(\chi(sl) - \zeta + cz)\big)\mathrm{sh}\big(m\chi(sl)\big)}\Bigg)\frac{\mathrm{sh}(\chi(wl))}{\mathrm{sh}((m+1)\chi(wl))}. \quad (12)$$

For a fixed $x > 0$ and $r = mc > 0$, as $c \to 0$:

$$\chi(xl) = \frac{c}{\sigma}\sqrt{2x + \frac{x^2c^2}{\sigma^2} + \frac{\nu^2}{\sigma^2}} + O(c^2) \to 0, \quad (13)$$

$$m\chi(xl) \to \alpha(x), \quad (14)$$

$$m(\zeta - cz) \to \varepsilon - rz, \quad (15)$$

$$\frac{\mathrm{sh}(\chi(xl))}{c} \to \frac{\alpha(x)}{r}. \quad (16)$$

Use the definition of $\mathrm{arccosh}(\cdot)$ and (13) can be written

$$\chi(xl) = \mathrm{arccosh}\left(\frac{1+xl}{\sqrt{4pq}}\right) = \ln\left(\frac{1+xl}{\sqrt{4pq}} + \sqrt{\left(\frac{1+xl}{\sqrt{4pq}}\right)^2 - 1}\right)$$
$$= \ln(1 + xl + \sqrt{(1+xl)^2 - 4pq}) - \ln(\sqrt{4pq}).$$

Express l, p and q in terms of c and simplify:

$$\ln\left(1 + \frac{xc^2}{\sigma^2} + \frac{c}{\sigma}\sqrt{2x + \frac{x^2c^2}{\sigma^2} + \frac{\nu^2}{\sigma^2}}\right) - \frac{1}{2}\ln\left(1 - \frac{c^2\nu^2}{\sigma^4}\right)$$

With Taylor expansion the expression above can be written as

$$\frac{xc^2}{\sigma^2}+\frac{c}{\sigma}\sqrt{2x+\frac{x^2c^2}{\sigma^2}+\frac{\nu^2}{\sigma^2}}+O(c^2)-\frac{1}{2}\Big(-\frac{c^2\nu^2}{\sigma^4}+O(c^4)\Big)$$

$$=\frac{c}{\sigma}\sqrt{2x+\frac{x^2c^2}{\sigma^2}+\frac{\nu^2}{\sigma^2}}+O(c^2)\to 0 \quad \text{as} \quad c\to 0, \tag{17}$$

proving (13). From this it follows that

$$m\chi(xl)=\frac{r}{\sigma}\sqrt{2x+\frac{x^2c^2}{\sigma^2}+\frac{\nu^2}{\sigma^2}}+O(c)\to\frac{r}{\sigma}\sqrt{2x+\frac{\nu^2}{\sigma^2}}=\alpha(x) \quad \text{as} \quad c\to 0,$$

and (14) is proved. Similarly we get for (15):

$$m(\zeta-cz)=m\Big(\ln\sqrt{\frac{p}{q}}-cz\Big)=\frac{r}{2c}\Big[\ln\Big(1+\frac{c\nu}{\sigma^2}\Big)-\ln\Big(1-\frac{c\nu}{\sigma^2}\Big)\Big]-rz$$

$$=\frac{r\nu}{\sigma^2}+O(c)-rz\to\frac{r\nu}{\sigma^2}-rz=\varepsilon-rz \quad \text{as} \quad c\to 0.$$

To prove (16) use the definition of $\sinh(\cdot)$,

$$\frac{\mathrm{sh}(\chi(xl))}{c}=\frac{1}{2c}(e^{\chi(xl)}-e^{-\chi(xl)})=\frac{e^{-\chi(xl)}}{2c}(e^{2\chi(xl)}-1)$$

$$=\frac{e^{-\chi(xl)}}{2c}(1+2\chi(xl)+O(c^2)-1)=\frac{e^{-\chi(xl)}}{c}(\chi(xl)+O(c^2)), \tag{18}$$

and substitute (17) in the last equation in (18),

$$e^{-\chi(xl)}\Big(\frac{1}{\sigma}\sqrt{2x+\frac{x^2c^2}{\sigma^2}+\frac{\nu^2}{\sigma^2}}+O(c)\Big)\to\frac{1}{\sigma}\sqrt{2x+\frac{\nu^2}{\sigma^2}}=\frac{\alpha(x)}{r} \quad \text{as} \quad c\to 0,$$

and (16) is proved. Using the limits (13), (14), (15) and (16) in (12), we find that (12) tends to

$$\left(\frac{\mathrm{sh}^2\big(\frac{\alpha(s)+\varepsilon-rz}{2}\big)}{(\alpha(s)+\varepsilon-rz)\mathrm{sh}(\alpha(s))}+\frac{\mathrm{sh}^2\big(\frac{\alpha(s)-\varepsilon+rz}{2}\big)}{(\alpha(s)-\varepsilon+rz)\mathrm{sh}(\alpha(s))}\right)\frac{2\alpha(w)}{\mathrm{sh}(\alpha(w))}$$

as $c\to 0$ and the assertion follows. $\square$

According to [2] Theorem 7 the above limit is the Laplace transform for $(U, T-U, W_T)$ where $\{W_t\}_{t\geq 0}$ is a Brownian motion, T is the time it takes for it to cover an interval of length r and U is the time it takes to reach the other endpoint for the first time. The random walk and the continuous time Markov chain converge to the same limit, a Brownian motion with drift ν and variance σ^2, the extra variation in the holding times does not matter. The density of $\frac{\sigma^2}{r^2}T$, $h_r(y)$ see (19), is derived in [2] by inverting the Laplace transform:

$$h_r(y) = 2e^{-\varepsilon^2 y/2} \sum_{k=1}^{\infty} \frac{(\pi k)^2}{\varepsilon^2 + (\pi k)^2} \Big[(\pi k)^2 (1 - (-1)^k \mathrm{ch}(\varepsilon)) y - \frac{3\varepsilon^2 + (\pi k)^2}{\varepsilon^2 + (\pi k)^2} (1 - (-1)^k \mathrm{ch}(\varepsilon)) - (-1)^k \varepsilon \mathrm{sh}(\varepsilon) \Big] e^{-(\pi k)^2 y/2}. \quad (19)$$

In Figure 4 and Figure 5 some examples are given of the densities of ΔN, (7), T_N, (4), and T, (19), as Δ, the time between the steps, and l, the expected time between the steps, decrease to zero. The solid line is the same in (a), (b) and (c), it is just the simple random walk and the birth and death process that are scaled. Table 1 gives the parameters that are used. Note that the convergence occurs when $m \to \infty$, $c \to 0$, $l \to 0$ and $p \to 1/2$ but it is enough to specify only one of the parameters, the rest are determined by the relations $p = \frac{1}{2}\big(1 + \frac{c\nu}{\sigma^2}\big)$, $c^2 = \sigma^2 l$, ($c^2 = \sigma^2 \Delta$), and $r = mc$.

In the same way as we scaled N and T_N above and let the the time between the steps decrease to zero we can scale the range covered a certain time t. Figure 6 and Figure 7 give examples of the density $g_t(r)$, see (20) below, of the range $R_{t,B}$ of a Brownian motion at time t and scaled R_n and R_t, (8). Note that the density of $R_{t,B}$ is the same in (a)-(f) it is just R_n and R_t that are scaled. Table 2 lists the parameters used in Figure 6 and Figure 7. In Chong, Cowan and Holst [2] the density of $R_{t,B}$ is given as:

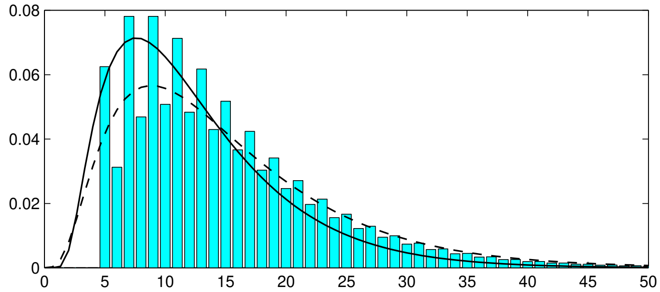
$$\begin{aligned} g_t(r) = {} & \frac{2t\sigma^2}{r^3} h_r\Big(\frac{t\sigma^2}{r^2}\Big) + 4\nu\sigma^4 e^{-t\nu^2/(2\sigma^2)} \sum_{k=1}^{\infty} \frac{(\pi k)^2 e^{-(\pi k \sigma)^2 t/(2r^2)}}{[(r\nu)^2 + (\pi k \sigma^2)^2]^2} \\ & \times \Bigg[r\nu \Big\{ \frac{12\sigma^4[(r\nu)^2 - (\pi k \sigma^2)^2]}{[(r\nu)^2 + (\pi k \sigma^2)^2]^2} + \frac{\sigma^2 t[3(r\nu)^2 - 5(\pi k\sigma^2)^2]}{r^2[(r\nu)^2 + (\pi k \sigma^2)^2]} \\ & - \frac{(\pi k \sigma^2 t)^2}{r^4} \Big\} \Big(1 - (-1)^k \mathrm{ch}\Big(\frac{r\nu}{\sigma^2}\Big)\Big) - r\nu(-1)^k \mathrm{ch}\Big(\frac{r\nu}{\sigma^2}\Big) \\ & + (-1)^k \mathrm{sh}\Big(\frac{r\nu}{\sigma^2}\Big) \Big\{ \frac{t((r\nu)^2 - (\pi k \sigma^2)^2)}{r^2} + \frac{2\sigma^2(3(r\nu)^2 - (\pi k\sigma^2)^2)}{(r\nu)^2 + (\pi k \sigma^2)^2} \Big\} \Bigg]. \quad (20) \end{aligned}$$

	m	c	l, Δ	p	Figure
$r = 5$, $\sigma^2 = 1$, $\nu = 0$	5	1	1	0.5	4(a)
	10	0.5	0.25	0.5	4(b)
	100	0.05	0.0025	0.5	4(c)
$r = 1$, $\sigma^2 = 1$, $\nu = 5$	10	0.1	0.01	0.75	5(a)
	20	0.05	0.0025	0.625	5(b)
	100	0.01	0.0001	0.525	5(c)

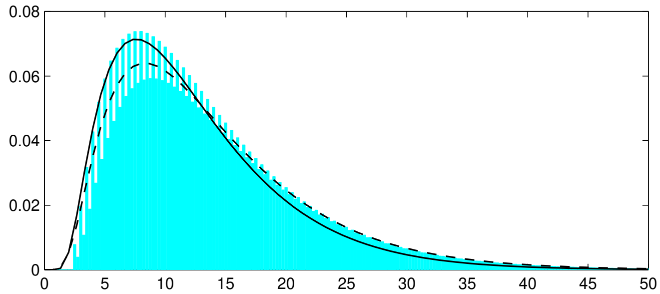
Table 1: Parameters in Figure 4 and Figure 5.

	c	l, Δ	p	Figure
$t = 1$, $\sigma^2 = 1$, $\nu = 1$	0.5	0.25	0.75	6(a)-(b)
	0.2	0.04	0.60	6(c)-(d)
	0.1	0.01	0.55	6(e)-(f)
$t = 1$, $\sigma^2 = 1$, $\nu = 3$	0.3	0.09	0.95	7(a)-(b)
	0.2	0.04	0.80	7(c)-(d)
	0.1	0.01	0.65	7(e)-(f)

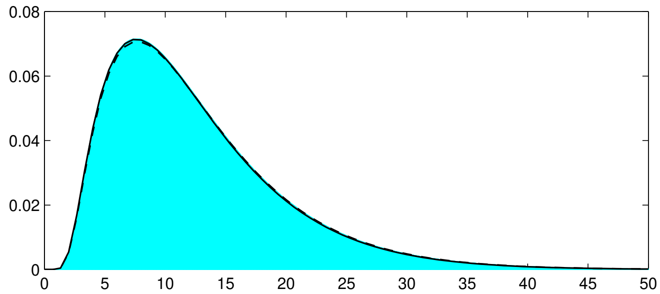
Table 2: Parameters in Figure 6 and Figure 7.



(a)



(b)



(c)

Figure 4: The solid line is the density of the cover time of an interval of length 5 for a Brownian motion with drift 0 and variance parameter 1. The bars and the dashed line are the density of the cover time of a scaled simple random walk respective a scaled birth and death process.

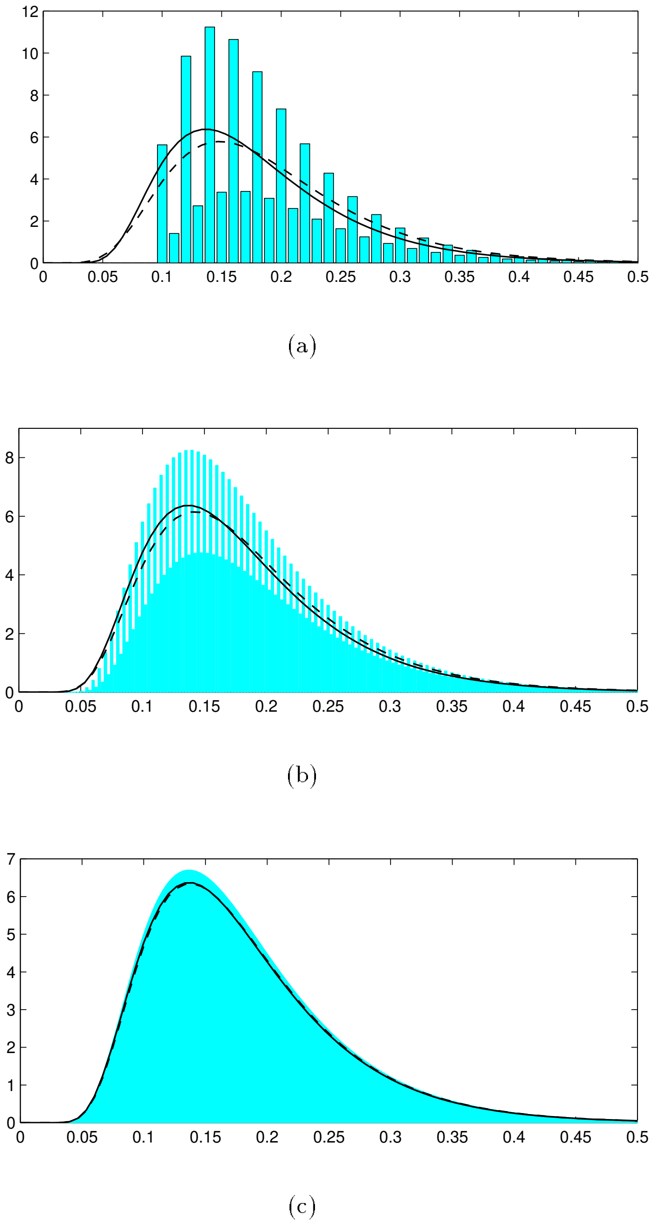
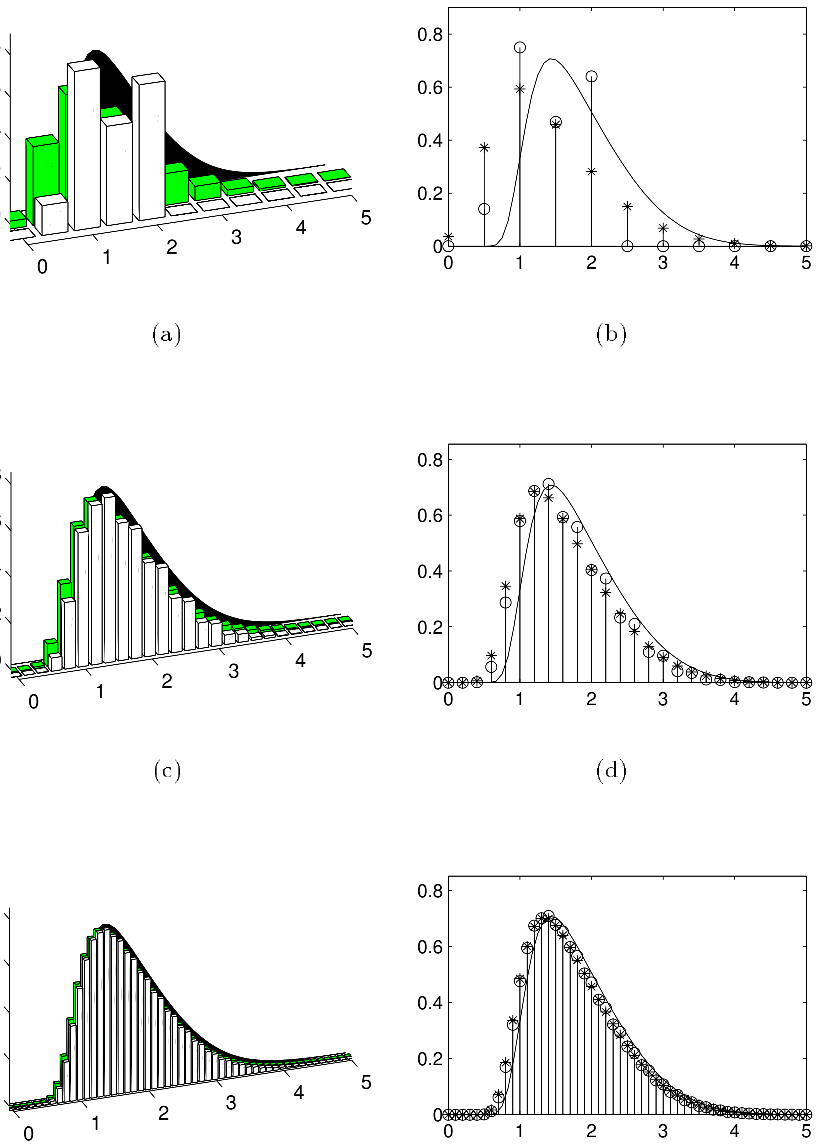


Figure 5: The solid line is the density of the cover time of an interval of length 1 for a Brownian motion with drift 5 and variance parameter 1. The bars and the dashed line are the density of the cover time of a scaled simple random walk respective a scaled birth and death process.



(a) (b)

(c) (d)

(e) (f)

Figure 6: The white bars and the circles are the density of R_n, the grey bars and the stars are the density of R_t and the black shape and the solid line are the density of $R_{t,B}$.

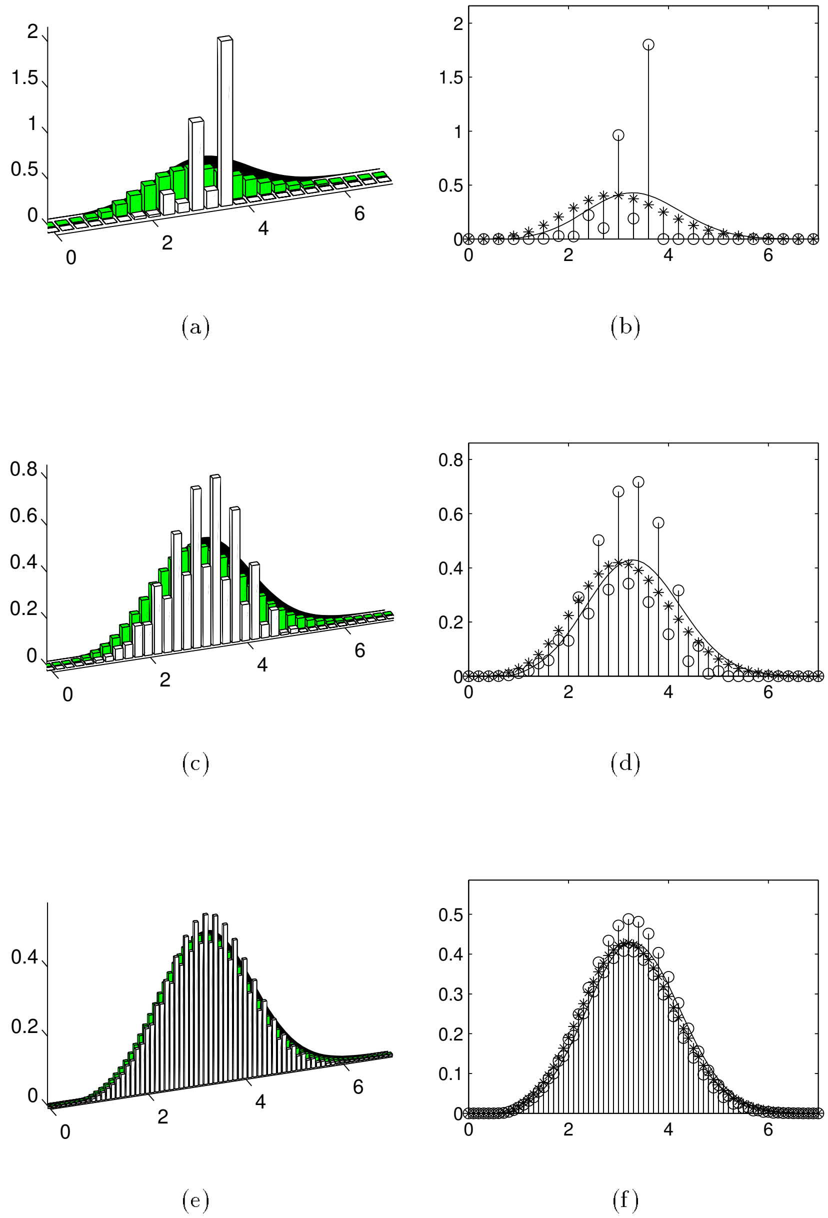


Figure 7: The white bars and the circles are the density of R_n, the grey bars and the stars are the density of R_t and the black shape and the solid line are the density of $R_{t,B}$.

5 Concluding remarks

The difference between the random walk approach and the continuous Markov chain approach is that the holding times are continuous and random. The irregularities between odd and even steps that appears for the random walk are smoothed out for the continuous Markov chain due to that continuous holding times are used. The calculations concludes that it does not affect the limit distribution if the holding times are constant or exponentially distributed i.e. the variation in the holding times does not influence the limit distribution. The approach to the limit is on the other hand influenced of the holding times. The distribution of the cover time for the continuous Markov chain is closer to the limit distribution and the approach to the limit is faster. The technique used for deriving the Laplace transform for the continuous time Markov chain does not depend on the distribution of the holding times. As long as the jump chain is independent of the holding times any distribution could be used. Restrictions on the distribution of general holding times would be required to obtain limit distributions.

Appendix

When there are long and complicated formulas, as in this paper, we think it is important to get an intuitive understanding. With a computer it is possible to get that by plotting the curves. It visualizes the convergence of the scaled densities and probability functions and can also be used to see if the calculations are reasonable. These are the reasons why we put much time and effort in these graphs. The graphs are done in Matlab. Therefore the matrix structure in the formulas are used as much as possible to speed up the computations. Below follows a short explanation how they are produced.

Figure 2

input m (the range)
input p (the probability to jump one step up)
input interval (the length of the interval to be plotted)

%calculation of P(N=n)
compute $T_m = \{\theta(k,m)\}_{k=1}^{m-1}$ and $T_{m+1} = \{\theta(k,m+1)\}_{k=1}^{m}$ in two matrices
compute $C_m = \left\{\cos\frac{\pi k}{m}\right\}_{k=1}^{m-1}$ and $C_{m+1} = \left\{\cos\frac{\pi k}{m+1}\right\}_{k=1}^{m}$ in two matrices
n=0:interval
$P = P(N=n) = (4pq)^{n/2}\left(T_m' C_m^{n-1} - T_{m+1}' C_{m+1}^{n-1}\right)$
    %Where C_m^{n-1} and C_{m+1}^{n-1} stands for the elements in the matrices raised to n-1.

%calculation $f_{T_N}(t)$
t=0:interval
compute $T_n = \left\{\frac{t^{n-1}e^{-t}}{\Gamma(n)}\right\}_{n\geq m}$ in a matrix for each t to be plotted
    %The terms becomes small fast, for a good plot just a few terms are necessary.
$f_{T_N}(t) = T_n' P(n \geq m)$

%produces the plots
plot(n,P(n))
plot(t,$f_{T_N}(t)$)

Figure 3

input t (the time)
input p (the probability to jump one step up)
input interval (the length of the interval to be plotted)

input tune

%R_t consists of an infinite sum which converges fast, for a good plot just a few terms are necessary, this parameter sets the number of terms.

compute $R_t(0)$ and $R_n(0) = 0$
compute $N_1 = \{P(N_1 = n)\}_{n=1}^{\text{tune}}$ in a matrix

%$P(N_1 = 1) = 1$ and $P(N_1 = n) = 0,\ n > 1$

compute $G = \{P(Y_n \leq t)\}_{n=1}^{\text{tune}}$ where $Y_n \sim \Gamma(n, 1)$ in a matrix

for m=1:interval
compute $N_{m+1} = \{P(N_{m+1} = n)\}_{n=m+1}^{\text{tune}}$ in a matrix
if $m \leq t$
compute $R_n(m) = N_m(m) + \text{sum}(N_m(m+1:t) - N_{m+1}(m+1:t)$
end
compute $R_t(m) = G(m)N_m(m) + G(m+1:\text{tune})$

$\times(N_m(m+1:\text{tune}) - N_{m+1}(m+1:\text{tune})'$

end

T=0:interval

plot(T,$R_n(n)$)
plot(T,$R_t(n)$)

Figure 4 and Figure 5

input r (the range)
input ν (the drift parameter of the Brownian motion)
input σ^2 (the variance parameter of the Brownian motion)
input c (the scaled length of the jumps)
$p = \frac{1}{2}\left(1 + \frac{c\nu}{\sigma^2}\right)$ (the probability to jump one step up)
$l = \frac{c^2}{\sigma^2}$ (the scaled mean length of the time steps)
input interval (the length of the interval to be plotted)
input tune

%R_t consists of an infinite sum which converges fast, for a good plot just a few terms are necessary, this parameter sets the number of terms.

input bmtune

%$R_{t,B}$ consists of an infinite sum which converges fast, for a good plot just a few terms are necessary, this parameter sets the number of terms.

%calculation of $P(N = n)$
compute $T_m = \{\theta(k, m)\}_{k=1}^{m-1}$ and $T_{m+1} = \{\theta(k, m+1)\}_{k=1}^{m}$ in two matrices

compute $C_m = \left\{\cos\frac{\pi k}{m}\right\}_{k=1}^{m-1}$ and $C_{m+1} = \left\{\cos\frac{\pi k}{m+1}\right\}_{k=1}^{m}$ in two matrices

%Where C_m^{n-1} and C_{m+1}^{n-1} stands for the elements in the matrices raised to n-1.

n=0:interval/l+tune

$P(N=n) = (4pq)^{n/2}\left(T_m' C_m^{n-1} - T_{m+1}' C_{m+1}^{n-1}\right)$

P(n)=P(N=n)/l

%calcualtion of $f_{T_N}(t)$

t=0:interval

compute $T_n = \left\{\frac{t^{n-1}e^{-t/l}}{\Gamma(n)l^n}\right\}_{n\geq m}$ in a matrix for each t to be plotted

%The terms become small fast, for a good plot just a few terms are necessary, use tune to model how many that are needed.

$f_{T_N}(t) = T_n' P(n \geq m)$

%calculation of $f_T(t)$

compute $f_T(t) = \{h_r(t)\}_{t=0}^{\text{interval}}$, equation (19), in a matrix

% $h_r(t)$ includes an infinite sum which converges fast, use bmtune to model how many terms should be included for a good plot.

plot(n,P(n))

plot(t,$f_{T_N}(t)$)

plot(t,$f_T(t)$)

Figure 6 and Figure 7

input ν (the drift of Brownian motion)

input σ^2 (the variance of the Brownian motion)

input t (the time)

input l (the scaled mean length of the time steps)

$c = \sqrt{\sigma^2 l}$ (the scaled length of the jumps)

$p = \frac{1}{2}\left(1 + \frac{c\nu}{\sigma^2}\right)$ (the probability to jump one step up)

input interval (the length of the interval to be plotted)

input tune

%R_t consists of an infinite sum which converges fast, for a good plot just a few terms are necessary, this parameter sets the number of terms.

input bmtune

%$R_{t,B}$ consists of an infinite sum which converges fast, for a good plot just a few terms are necessary, this parameter sets the number of terms.

%calculation of the scale for the range for R_t and R_n

rd=0:c:interval

compute $R_t(0)$ and $R_n(0) = 0$
compute $N_1 = \{P(N_1 = n)\}_{n=1}^{\text{tune}}$ in a matrix
    $\% P(N_1 = 1) = 1$ and $P(N_1 = n) = 0,\ n > 1$
compute $G = \{P(Y_n \le t)\}_{n=1}^{\text{tune}}$ where $Y_n \sim \Gamma(n, l)$ in a matrix

%calculation of $R_t(rd)$ and $R_n(rd)$
for m=1:interval/c
compute $N_{m+1} = \{P(N_{m+1} = n)\}_{n=m+1}^{\text{tune}}$ in a matrix
if $m \le \frac{t}{l}$
compute $R_n(m) = N_m(m) + \text{sum}(N_m(m+1:t) - N_{m+1}(m+1:t)$
end
compute $R_t(m) = G(m)N_m(m) + G(m+1:\text{tune})$
    $\times (N_m(m+1:\text{tune}) - N_{m+1}(m+1:\text{tune})'$
end

%calculation of the range of $R_{t,B}(r)$
r=0:interval

%calculation of $R_{t,B}(r)$
compute $R_{t,B}(r) = \{g_t(r)\}_{r=0}^{\text{interval}}$, equation (20), in a matrix
    $\% g_t(r)$ includes an infinite sum which converges fast, use bmtune to model how many terms should be included for a good plot.

plot(rd,$R_n(rd)$)
plot(rd,$R_t(rd)$)
plot(r,$R_{t,B}(r)$)

References

[1] Norris, J.R. (1997) *Markov Chains.* Cambridge University Press, Cambridge.

[2] Chong, K.S., Cowan, R. and Holst, L. (2000) The ruin problem and cover times of asymmetric random walks and Brownian motions. *Adv. Appl. Prob.* **32**, 177-192.