Asymptotic theory for a risk process with a high dividend barrier

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Abstract

The only way to avoid ruin in the classical model of the collective risk theory is that the surplus increases to infinity. We consider a modified model with a dividend barrier which prevents this behavior. It is shown that there is a simple approximation formula for the time of ruin when the level of the dividend barrier is high. A numerical example is presented in the case when the claims are exponentially distributed. The relation to queuing theory is used to derive the proportion of time the surplus is below some given level.

keywords: Collective risk theory; time of ruin; dividend barrier; Wald's fundamental identity; Legendre transform; M/G/1-queue.

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1 Introduction

In the classical collective risk theory, created by Lundberg [11], a risk process

$$X(t) = u + pt - S_t \tag{1}$$

is used to model the surplus in an insurance company, where u is the initial surplus and $X(t)$ is the surplus at time t, see Fig. 1. The positive constant p is the premium income rate and $\{S_t\}$ denotes aggregate claims up to time t. It is assumed that $\{S_t\}$ is a compound Poisson process with Poisson parameter λ and distribution function F of claim amounts. Ruin occurs if the surplus becomes negative for some t.

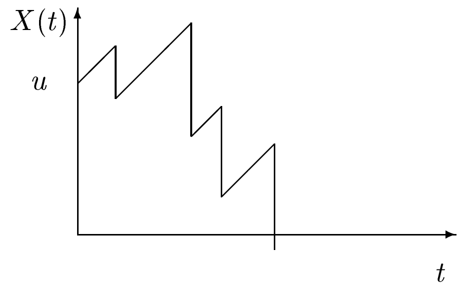


Figure 1: $X(t) = u + pt - S_t$.

An important quantity is the safety loading $\frac{p}{\lambda\mu} - 1$, where μ is the expected size of a claim. When the risk process has positive safety loading, i.e. the risk process has positive drift, the ruin probability is less than one and given by

$$\psi(u) = P\big(\inf_{0\le t<\infty} X(t) < 0 \mid X(0) = u\big) = \big(1 - \frac{\lambda\mu}{p}\big) \sum_{n=0}^{\infty} \big(\frac{\lambda\mu}{p}\big)^n F_T^{n*}(u), \tag{2}$$

where $F_T(x)$ is the integrated tail distribution

$$F_T(u) = \frac{1}{\mu} \int_0^u \big(1 - F(y)\big)\, dy.$$

This formula is sometimes called *Beekman's convolution formula* and corresponds to the *Pollaczek-Khintchine formula* in queueing theory. Under certain conditions (the *Cramér-Lundberg condition*) for the claim distribution, it is possible to transform the defective renewal function (2) into a proper renewal function. An application of the key renewal theorem yields the well-known *Cramér-Lundberg approximation*, see Embrechts, Klüppelberg and Mikosch [9],

$$\psi(u) \sim Ce^{-Ru} \quad \text{as } u \to \infty, \tag{3}$$

where C is a constant depending on the parameters in the model and R is sometimes called the *Lundberg exponent* or the *adjustment coefficient*, see also Lundberg [12] and Cramér [6]. The only way to avoid ruin is that the surplus increases to infinity. A more realistic model is obtained if the risk process in between claims is allowed to vary deterministically but not necessarily linearly. For example, if the surplus is invested at an certain interest rate or all earnings above a given level are paid out as dividend or used for other purposes. We will assume that the risk process satisfies the *storage equation*, see Prabhu [14],

$$X(t) = u + \int_0^t p\big(X(s)\big)ds - S_t, \tag{4}$$

i.e. the premium income rate depends on the current surplus. The ruin probability in this type of model, when the risk process still has a chance of increasing to infinity, has been studied by Asmussen and Schock Petersen [4], Davidsson [7], Djehiche [8], Gerber [10] and many others. We will consider the risk process for the following special choice of the premium income function

$$p(x) = pI_{(x \le b)}.$$

The level b is a dividend barrier. Below the dividend barrier, no dividends are paid out. On the dividend barrier, a dividend at rate p is paid out until the next claim, see Figure 2. Ruin will occur with probability one and we want to know when it occurs.

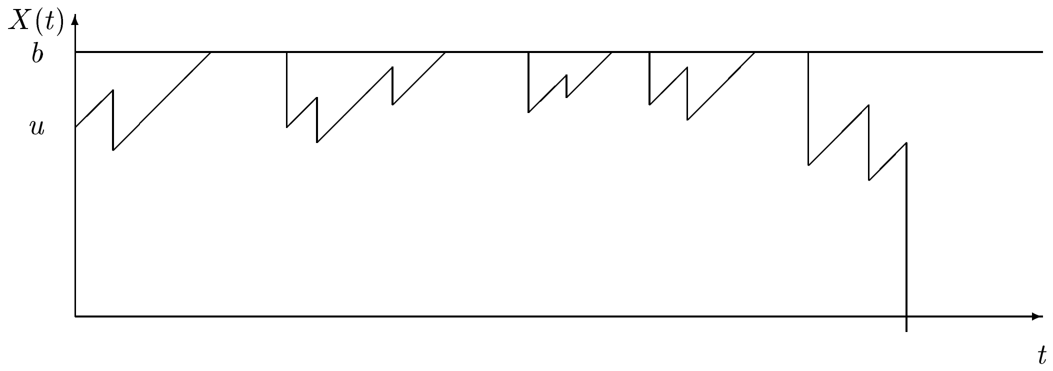


Figure 2: $X(t) = u + p\int_0^t I_{(X(s)\le b)}ds - S_t$.

Segerdahl [18] has studied this problem with the technique of integral-differential equations and gives explicitly the characteristic function of the time of ruin in the case when the claims are exponentially distributed. We will use another technique to show that there is, for large b, a simple approximation formula for the distribution of the time of ruin. There are two different cases:

(i) An initial surplus just below a high dividend barrier: The risk process will move towards the dividend barrier and visit it a large geometric number of times before ruin. Ruin will occur after a very long exponentially distributed time. This case is analyzed in section 2.

(ii) An initial surplus much lower than a high dividend barrier: Either ruin will occur quite soon without visit to the dividend barrier or the risk process will reach the dividend barrier and ruin occurs according to the exponential distribution in (i). This case is analyzed in section 3.

It is not only interesting to know the time of ruin, it is also interesting to know the proportion of time the surplus is below some given level. This is especially interesting in the case when the risk process reaches the high dividend barrier. In section 4, the connection between risk theory and queueing theory is used to derive this proportion. In section 5, a numerical example is presented in the case when the claims are exponentially distributed. The Laplace transform of the time of ruin is calculated and numerically inverted to obtain the density. The obtained density is compared with the density given by the approximation formula in section 3.

2 An initial surplus just below a high dividend barrier

The risk process to be considered in this chapter is supposed to satisfy equation (4) with $u = b - x$. In other words, suppose that

$$X(t) = b - x + p \int_0^t I_{(X(s) \leq b)} ds - S_t,$$

where S_t is a compound Poisson process defined by its moment generating function

$$E\left[e^{zS_t}\right] = e^{t\lambda \int_0^\infty (e^{zx}-1)F(dx)} \equiv e^{tg(z)}. \tag{5}$$

The risk process starts at $u = b - x$ and moves like the classical risk process (1) except that it will never increase above the dividend barrier at the level b. Let the time of ruin be

$$T_R = \inf\{t : X(t) < 0\}.$$

We want to derive an asymptotic formula for

$$P_{b-x}(T_R > t) = P(T_R > t \mid X(0) = b - x),$$

when b is large and $x \ll b$. Throughout this chapter you should think of b as large and x very small compared with b, i.e. the initial surplus is close to the high dividend barrier. Assume that:

(i) the safety loading is positive, i.e. $\frac{p}{\lambda\mu} - 1 > 0$ or $p > \lambda\mu = g'(0)$,

(ii) the *Cramér-Lundberg condition* is satisfied for some $R > 0$, i.e. $\exists R > 0$ such that $g(R) = pR$ and $g'(R) < \infty$.

The last assumption implies by Markov's inequality applied to e^{Rx}

$$1 - F(x) \leq \big(\frac{pR}{\lambda} + 1\big)e^{-Rx}, \tag{6}$$

i.e. the probability of a large claim is very small. At the start the risk process will, with a very high probability, move towards the dividend barrier and perform a recurrent motion in the vicinity of it. The path can be partitioned into excursions between successive visits to the dividend barrier. Denote the time of visit number k to the dividend barrier by T_k, see Figure 3.

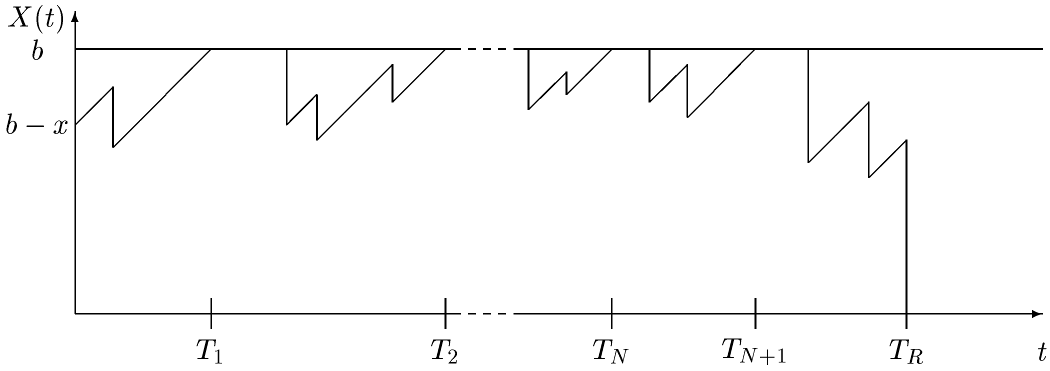


Figure 3: $X(t) = b - x + p\int_0^t I_{(X(s)\leq b)}ds - S_t$,

By the strong Markov property, the lengths of the completed excursions $\{T_{k+1} - T_k\}_{k\geq 1}$ are independent identically distributed random variables. During each excursion there is a small probability

$$\rho = P\big(T_{k+1} - T_k = +\infty \mid T_k < \infty\big)$$

that ruin will occur. The visit times $\{T_k\}$ form a defective renewal process and the number of completed excursions before ruin has a geometric distribution

$$P(N = k) = (1-\rho)^k\rho \quad k = 0, 1, 2, \dots \quad . \tag{7}$$

The time of ruin can be written as a random sum of independent excursions

$$T_R = T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}). \tag{8}$$

Since the probability of ruin during an excursion is small the number of completed excursions before ruin will be large. Hence by the law of large numbers

$$\sum_{k=1}^{N}(T_{k+1}-T_k) \overset{d}{\approx} Nm,$$

where

$$m = E\big[T_{k+1}-T_k \mid T_{k+1}<\infty\big]$$

is the expected length of a complete excursion. If T_1 and $\{T_R - T_{N+1}\}$ are small in probability compared with the random sum $\sum_{k=1}^{N}(T_{k+1}-T_k)$, then Slutsky's theorem yields

$$P_{b-x}(T_R > t) \approx P_{b-x}(N > \frac{t}{m}) \approx (1-\rho)^{[\frac{t}{m}]} \approx e^{-\frac{t\rho}{m}}.$$

The technique described above has also been used by Andersson and Djehiche [2] in the context of a birth and death process. We have the following theorem which will be proved more rigorously below.

Theorem 2.1 *The time of ruin has asymptotically an exponential distribution with expected value* $\frac{m}{\rho}$. *In other words,*

$$P_{b-x}\big(T_R > t\big) \sim e^{-\frac{\rho t}{m}} \quad \textit{as } b \to \infty.$$

The expected length of a complete excursion satisfies

$$m \sim \frac{1}{\lambda} + \frac{\mu}{p-\lambda\mu} \quad \textit{as } b \to \infty$$

and the probability of ruin during an excursion satisfies

$$\rho \sim \frac{CpR}{\lambda} e^{-Rb} \quad \textit{as } b \to \infty,$$

where R *is the* Lundberg exponent *or the* adjustment coefficient, *see assumption (ii), and* $C = \frac{p-g'(0)}{g'(R)-p}$ *is the constant in the* Cramér-Lundberg approximation *(3).*

The proof of the theorem will follow from the lemmas below. Let

$$\psi(b-x, b) = P\big(T_R < T_1 \mid X(0) = b-x\big),$$

i.e. the probability of ruin without visit to the dividend barrier given that $u = b-x$.

Lemma 2.1 *The probability of ruin without visit to the dividend barrier is given by*

$$\psi(b-x,b) = \frac{\psi(b-x)-\psi(b)}{1-\psi(b)},$$

where $\psi(x)$ is the ruin probability in the classical model (1) (without dividend barrier) and given by (2). Moreover, when x is fixed

$$\psi(b-x,b) \sim Ce^{-Rb}(e^{Rx}-1) \quad \text{as } b \to \infty.$$

Proof. By the strong Markov property and since $min(T_1, T_R) < \infty$ a.s., it follows that, see Rogers [15],

$$\begin{aligned} 1-\psi(b-x) &= P_{b-x}(\text{no ruin without barrier}) \\ &= P_{b-x}(\text{no ruin before visit to b}) P_b(\text{no ruin without barrier}) \\ &= \big(1-\psi(b-x,b)\big)\big(1-\psi(b)\big), \end{aligned}$$

which yields

$$\psi(b-x,b) = \frac{\psi(b-x)-\psi(b)}{1-\psi(b)}.$$

Apply the *Cramér-Lundberg approximation* (3) and the proof is complete. □

We see that the risk process will indeed, with a probability almost equal to 1, reach the dividend barrier and the distribution of the time of ruin is determined by the random sum (8). The next step is to show that $\sum_{k=1}^{N}(T_{k+1}-T_k)$ is asymptotically exponentially distributed. We begin by calculating the probability of ruin during an excursion, which is the parameter in the geometric distribution (7) of the number of completed excursions.

Lemma 2.2 *The probability of ruin during an excursion satisfies*

$$\rho \sim \frac{CpR}{\lambda} e^{-Rb} \quad \text{as } b \to \infty,$$

where R is the Lundberg exponent *or the* adjustment coefficient *and* $C = \frac{p-g'(0)}{g'(R)-p}$ *is the constant in the* Cramér-Lundberg approximation *(3).*

Proof. Integrate $\psi(b-x,b)$ with respect to the claim distribution and split the integral in the following way in order to use Lemma 2.1. Take a small $\varepsilon > 0$,

$$\begin{aligned}\rho &= \int_0^b \psi(b-x,b)F(dx) + \int_b^\infty F(dx) = \int_0^\infty \psi(b-x,b)F(dx) \\ &= \int_0^{b(1-\varepsilon)} \psi(b-x,b)F(dx) + \int_{b(1-\varepsilon)}^\infty \psi(b-x,b)F(dx). \end{aligned} \tag{9}$$

In the integral $\int_0^{b(1-\varepsilon)} \psi(b-x,b)F(dx)$ we have $b\varepsilon \le b-x \le b$. Hence, by Lemma 2.1,

$$\int_0^{b(1-\varepsilon)} \psi(b-x,b)F(dx) \sim Ce^{-Rb} \int_0^{b(1-\varepsilon)} (e^{Rx}-1)F(dx) \quad \text{as } b \to \infty.$$

The *Cramér-Lundberg condition* $\int_0^\infty (e^{Rx}-1)F(dx) = \frac{pR}{\lambda}$, i.e. the integral is convergent, yields

$$Ce^{-Rb} \int_0^{b(1-\varepsilon)} (e^{Rx}-1)F(dx) \sim Ce^{-Rb} \int_0^\infty (e^{Rx}-1)F(dx) = \frac{CpR}{\lambda} e^{-Rb}.$$

We have the following estimate of the second integral in (9)

$$\begin{aligned}\int_{b(1-\varepsilon)}^\infty \psi(b-x,b)F(dx) &\le \int_{b(1-\varepsilon)}^\infty F(dx) = \int_{b(1-\varepsilon)}^\infty e^{-Rx(1+2\varepsilon)} e^{Rx(1+2\varepsilon)} F(dx) \\ &\le e^{-Rb(1-\varepsilon)(1+2\varepsilon)} \big(\frac{g(R(1+2\varepsilon))}{\lambda} + 1 \big)\end{aligned}$$

and since ε is small $R(1-\varepsilon)(1+2\varepsilon) > R$. Hence the integral $\int_{b(1-\varepsilon)}^\infty \psi(b-x,b)F(dx)$ will be very small compared with $\frac{CpR}{\lambda}e^{-Rb}$ as $b \to \infty$ and the proof is complete. $\square$

We need some preparation to continue the proof of the theorem. Denote the *Esscher transform* of P by $P^{(z)}$:

$$\frac{dP^{(z)}}{dP} = e^{zS_t - tg(z)},$$

where $M_z(t) = e^{zS_t - tg(z)}$ can be recognized as *Wald's martingale*. Let T_f be the first passage time out from the interval $(0,b)$, i.e. the stopping time

$$T_f = \min\big(T_1, T_R\big),$$

then the *fundamental identity of sequential analysis*, sometimes called *Wald's fundamental identity*, implies that, see Asmussen [3],

$$P^{(z)}(T_f < \infty) = E\big[e^{zS_{T_f} - T_f g(z)} : T_f < \infty\big]. \tag{10}$$

Under the new measure $P^{(z)}$, the moment generating function of S_t is given by

$$E^{(z)}\left[e^{yS_t}\right] = e^{t(g(z+y)-g(z))}.$$

S_t is again a compound Poisson process and the risk process has the new drift $p - g'(z)$. From the theory of random walks it is well-known that the first passage time T_f is finite almost surely, hence the relation (10) yields

$$\begin{aligned} 1 &= E\left[e^{zS_{T_f} - T_f g(z)}\right] \quad \textit{or} \\ 1 &= E_{b-x}\left[e^{z(b-x-X(T_f)) - T_f(g(z)-pz)}\right], \end{aligned} \tag{11}$$

where the index $b-x$ indicates that the initial surplus is $u = b-x$. By differentiating it is easy to see that the function $f(z) = g(z) - pz$, appearing in equation (11), has the following properties:

(i) $f(z)$ is strictly convex,

(ii) $f(0) = f(R) = 0$,

(iii) $f(z)$ takes its minimum at the point z_p satisfying $g'(z_p) = p$.

Set

$$s = g(z) - pz, \quad s > f(z_p). \tag{12}$$

The properties of $f(z)$ implies that this equation has two different roots, $z_0(s) < z_1(s)$, see Figure 4. Hence the relation (11) can be written

$$1 = E_{b-x}\left[e^{z_{0,1}(s)(b-x-X(T_f)) - sT_f}\right], \quad s > h(z_p), \tag{13}$$

which is the key to the remaining proofs in this section. We will use the technique of Laplace transforms to derive the asymptotic distribution of the random sum $\sum_{k=1}^{N}(T_{k+1} - T_k)$. The next lemma gives the Laplace transform of the time to the first visit, or next visit, to the dividend barrier from a predetermined level "close" to b.

Lemma 2.3 *Conditional on the risk process reaching the dividend barrier, the Laplace transform of the time to the first visit to the dividend barrier satisfies*

$$E_{b-x}\left[e^{-sT_1} \mid \{T_1 < T_R\}\right] \sim E_{b-x}\left[e^{-sT_1} I_{\{T_1 < T_R\}}\right] \sim e^{xz_0(s)} \quad \textit{as } b \to \infty,$$

where $z_0(s)$ is the smallest root of equation (12).

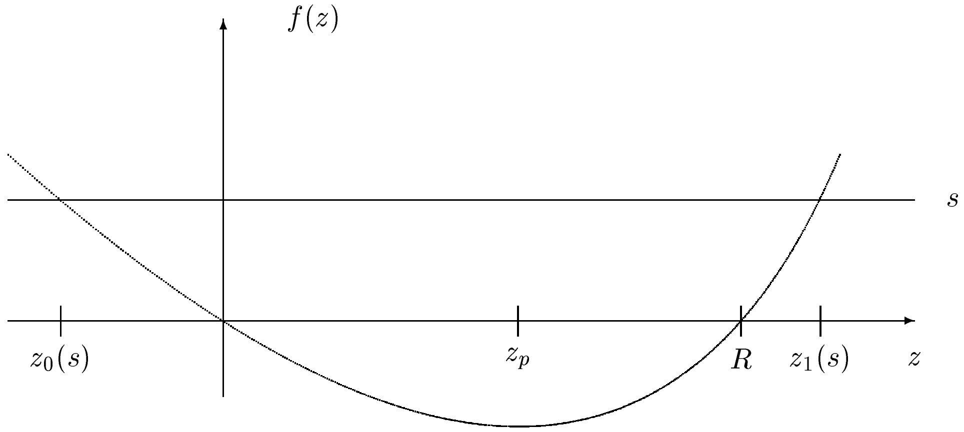


Figure 4: $f(z) = g(z) - pz$.

Proof. Equation (13) with the root $z_0(s)$ yields

$$\begin{aligned} 1 &= E_{b-x}\big[e^{z_0(s)(b-x-X(T_f))-sT_f}\big] \\ &= e^{-xz_0(s)}E_{b-x}\big[e^{-sT_1}I_{\{T_1<T_R\}}\big]+e^{z_0(s)(b-x)}E_{b-x}\big[e^{-z_0(s)X(T_R)-sT_R}I_{\{T_R<T_1\}}\big]. \end{aligned}$$

Take $s > 0$, then $z_0(s) < 0$ and $-z_0(s)X(T_R) \le 0$ on $\{T_R < T_1\}$. Hence

$$\begin{aligned} \big|E_{b-x}\big[e^{-sT_1}I_{\{T_1<T_R\}}\big]-e^{xz_0(s)}\big| &= e^{z_0(s)b}E_{b-x}\big[e^{-z_0(s)X(T_R)-sT_R}I_{\{T_R<T_1\}}\big] \\ &\le e^{z_0(s)b}P_{b-x}\big(T_R < T_1\big)\to 0 \quad \text{as } b\to\infty, \end{aligned}$$

where, by Lemma 2.1, $P_{b-x}\big(T_1 < T_R\big) \to 1$ as $b \to \infty$ and the proof is complete. □

Each complete excursion can be decomposed in an exponentially distributed holding time at the dividend barrier and a random distributed time from the risk process leaving the dividend barrier until the next visit. These two times are independent and the holding time at the barrier has an exponential distribution with expected value $\frac{1}{\lambda}$.

Lemma 2.4 *The Laplace transform of the length of a complete excursion satisfies*

$$E\big[e^{-s(T_{k+1}-T_k)} \mid T_{k+1} < \infty\big] \sim \frac{g(z_0(s)) + \lambda}{\lambda + s} \quad \textit{as } b \to \infty,$$

and the expected length of a complete excursion satisfies

$$m \sim \frac{1}{\lambda} + \frac{\mu}{p - \lambda\mu} \quad \textit{as } b \to \infty.$$

Proof. The Laplace transform of the exponentially distributed holding time at the barrier is $\frac{\lambda}{\lambda+s}$. By independence, since the probability of ruin during an excursion is small by Lemma 2.2 and since the probability of a large jump out from the dividend barrier is small by the *Cramér-Lundberg condition*, see (6), it follows by Lemma 2.3 that

$$E[e^{-s(T_{k+1}-T_k)} \mid T_{k+1} < \infty] \sim \frac{\lambda}{\lambda+s}\int_0^\infty e^{xz_0(s)}F(dx) \quad \text{as } b \to \infty.$$

The definition of $g(z)$ yields

$$E[e^{-s(T_{k+1}-T_k)} \mid T_{k+1} < \infty] \sim \frac{g(z_0(s))+\lambda}{\lambda+s} \quad \text{as } b \to \infty.$$

Differentiate to find that

$$\frac{d}{ds}E[e^{-s(T_{k+1}-T_k)} |T_{k+1} < \infty] \sim \frac{g'\big(z_0(s)\big)z_0'(s)(\lambda+s) - (g(z_0(s))+\lambda)}{(\lambda+s)^2} \quad \text{as } b \to \infty.$$

Since $s = g(z) - pz$,

$$z_0'(s) = -\frac{1}{p - g'\big(z_0(s)\big)}. \tag{14}$$

Set $s = 0$ and the proof is complete. □

The number N of completed excursions before ruin is geometrically distributed with expected value $\frac{1-\rho}{\rho}$, where ρ is given by Lemma 2.2. The lengths of the completed excursions are independent of each other and independent of N. We have the following lemma:

Lemma 2.5 *The Laplace transform of the total length of completed excursions satisfies*

$$E[e^{-s\frac{\rho}{m}\sum_{k=1}^N (T_{k+1}-T_k)}] \sim \frac{1}{1+s} \quad \textit{as } b{\to}\infty.$$

In other words, $\frac{\rho}{m}\sum_{k=1}^N (T_{k+1}-T_k)$ *is asymptotically exponentially distributed with expected value* 1.

Proof. The geometrically distributed random variable N has generating function

$$g_N(t) = E[t^N] = \frac{\rho}{1 - t(1-\rho)}.$$

Independence together with Lemma 2.4 yields

$$E[e^{-s\frac{\rho}{m}\sum_{k=1}^{N}(T_k - T_{k-1})}] \sim \frac{\rho}{1 - \frac{g(z_0(\frac{s\rho}{m}))+\lambda}{\lambda + \frac{s\rho}{m}}(1-\rho)} \quad \text{as } b \to \infty.$$

We have that

$$g(z_0(\frac{s\rho}{m})) \sim g'(0) z_0'(0) \frac{s\rho}{m} = -\frac{\lambda\mu s\rho}{(p - \lambda\mu)m} \quad \text{as } b \to \infty,$$

where $z_0'(s)$ is given by (14). Therefore

$$E[e^{-s\frac{\rho}{m}\sum_{k=1}^{N}(T_k - T_{k-1})}] \sim \frac{1}{1+s} \quad \text{as } b {\to} \infty,$$

and the proof is complete. □

We will now give exponential bounds for the time to the first visit to the dividend barrier and for the time from the last visit until ruin. These bounds are given in terms of the Legendre transform of $g(z)$, the same method has been used by Martin-Löf [13] to estimate the time of ruin without dividend barrier. Define $h(x)$ to be the Legendre transform of $g(z)$. In other words

$$h(x) = \max_z (xz - g(z)).$$

The Legendre transform has the following properties, see Martin-Löf [13]:

(i) $h(x)$ is strictly convex,

(ii) $h(x) \geq 0$ and $h\big(g'(0)\big) = h(\lambda\mu) = 0$,

(iii) $th(p + \frac{1}{t}) \geq R$ and strictly convex for $t > 0$, with equality for $\bar{t} = \frac{1}{g'(R) - p}$.

The connection between the quantity $\bar{t}$ and the time of ruin in the classical model is described in section 3.

Lemma 2.6 *The time to the first visit to the dividend barrier has the following bound. If* $t > \frac{x}{p - g'(0)}$ *then*

$$P_{b-x}\big(T_1 > t \mid \{T_1 < T_R\}\big) \leq \frac{e^{-th(p - \frac{x}{t})}}{1 - \psi(b - x, b)}$$

and $h(p - \frac{x}{t}) > 0$. *Moreover,* $h(p - \frac{x}{t}) \uparrow h(p) > 0$ *as* $t \to \infty$.

Proof. Equation (13) with the root $z_0(s)$ yields

$$1 = E_{b-x}[e^{z_0(s)(b-x-X(T_1))-sT_1} I_{\{T_1<T_R\}}] + E_{b-x}[e^{z_0(s)(b-x-X(T_R))-sT_R} I_{\{T_R<T_1\}}].$$

Since $X(T_1) = b$ on $\{T_1 < T_R\}$, it follows that

$$E_{b-x}[e^{-sT_1} I_{\{T_1<T_R\}}] \leq e^{z_0(s)x}.$$

Take $s < 0$ such that $0 < z_0(s) < z_p$, see Figure 4, then

$$e^{-st} P_{b-x}\big(\{T_1 > t\} \cap \{T_1 < T_R\}\big) \leq e^{z_0(s)x}$$

or, with $s = g(z) - pz$,

$$P_{b-x}\big(\{T_1 > t\} \cap \{T_1 < T_R\}\big) \leq e^{-t((p-\frac{x}{t})z-g(z))}. \tag{15}$$

The exponent is optimized by the z-value satisfying $g'(z) = p - \frac{x}{t}$. $g'(z)$ is strictly increasing and hence the optimal z-value is positive for $p - \frac{x}{t} > g'(0)$, i.e. for $t > \frac{x}{p-g'(0)}$. The properties of the Legendre transform $h(x)$ yields

$$t \max_z \big((p - \frac{x}{t})z - g(z)\big) = th(p - \frac{x}{t}) > th\big(g'(0)\big) = 0, \quad t > \frac{x}{p - g'(0)}.$$

The optimal z-value tends to z_p, $g'(z_p) = p$, as $t \to \infty$ and therefore

$$\max_z \big((p - \frac{x}{t})z - g(z)\big) = h(p - \frac{x}{t})\uparrow(pz_p - g(z_p)) = h(p), \quad \text{as } t \to \infty.$$

Divide (15) by $P_{b-x}(T_1 < T_R) = 1 - \psi(b - x, b)$ and the proof is complete. □

By Lemma 2.6 and since, by Lemma 2.2, $\rho \sim \frac{CpR}{\lambda} e^{-Rb}$ as $b \to \infty$, it follows, for fixed $\varepsilon > 0$, that

$$P_{b-x}\big(\frac{\rho T_1}{m} > \varepsilon \mid \{T_1 < T_R\}\big) \leq \frac{e^{-\frac{m\varepsilon}{\rho} h(p - \frac{\rho x}{m\varepsilon})}}{1 - \psi(b - x, b)} \to 0 \quad \text{as } b \to \infty.$$

Hence the time to the first visit will be small in probability compared with the sum $\sum_{k=1}^{N}(T_{k+1} - T_k)$, which is of order $\frac{m}{\rho}$. The next lemma estimates the time from the last visit until ruin. The proof is almost the same as above.

Lemma 2.7 *Let T_L be the time from the risk process leaving the dividend barrier for the last time until ruin. If $t > \bar{t}$ then*

$$P(T_L > bt) \lesssim \frac{\lambda}{CpR}\Big(\frac{g(th(p + \frac{1}{t}))}{\lambda} + 1\Big)e^{-b(th(p+\frac{1}{t})-R)} \quad \textit{as } b \to \infty$$

and $th(p + \frac{1}{t}) > R$.

Proof. Equation (13) with the root $z_1(s)$ yields

$$1 = E_{b-x}\big[e^{z_1(s)(b-x-X(T_1))-sT_1} I_{\{T_1<T_R\}}\big] + E_{b-x}\big[e^{z_1(s)(b-x-X(T_R))-sT_R} I_{\{T_R<T_1\}}\big].$$

Since $z_1(s)X(T_R) \le 0$ on $\{T_R < T_1\}$, it follows that

$$E_{b-x}\big[e^{-sT_R} I_{\{T_R<T_1\}}\big] \le E_{b-x}\big[e^{-z_1(s)X(T_R)-sT_R} I_{\{T_R<T_1\}}\big] \le e^{-(b-x)z_1(s)}.$$

Take $s < 0$ such that $z_p < z_1(s) < R$, see Figure 4, then

$$e^{-s(b-x)t} P_{b-x}\big(\{T_R > (b-x)t\} \cap \{T_R < T_1\}\big) \le e^{-(b-x)z_1(s)}$$

or, with $s = g(z) - pz$,

$$P_{b-x}\big(\{T_R > (b-x)t\} \cap \{T_R < T_1\}\big) \le e^{-(b-x)t((p+\frac{1}{t})z-g(z))}.$$

Optimizing the exponent and using the properties of $h(x)$ in the same way as in the proof Lemma 2.6 yields

$$t \max_z \big((p + \frac{1}{t})z - g(z)\big) = th(p + \frac{1}{t}) > R, \quad t > \bar{t}.$$

Since $P_{b-x}\big(\{T_R > bt\} \cap \{T_R < T_1\}\big) \le P_{b-x}\big(\{T_R > (b-x)t\} \cap \{T_R < T_1\}\big)$, it follows that

$$P_{b-x}\big(\{T_R > bt\} \cap \{T_R < T_1\}\big) \le e^{-(b-x)th(p+\frac{1}{t})}, \quad t > \bar{t}.$$

Integrating with respect to the claim distribution and using the definition of $g(z)$ yields

$$P_b\big(\{T_L > bt\} \cap \{T_R < T_1\}\big) \le e^{-bth(p+\frac{1}{t})}\big(\frac{g(th(p + \frac{1}{t}))}{\lambda} + 1\big),$$

where $\{T_R < T_1\}$ is the event that ruin occurs before next visit to the dividend barrier. Divide by the probability of ruin during an excursion, given by Lemma 2.2, and the proof is complete. □

Since, by Lemma 2.2, $\rho \sim \frac{CpR}{\lambda} e^{-Rb}$ as $b \to \infty$, it follows, for fixed $t > \bar{t}$ and $\varepsilon > 0$, that

$$P_{b-x}\big(\frac{\rho T_L}{m} > \varepsilon\big) < P(T_L > bt) \quad \text{as } b \to \infty,$$

where the expected length m of an excursion is given by Lemma 2.4. Therefore, by Lemma 2.7,

$$P_{b-x}\big(\frac{\rho T_L}{m} > \varepsilon\big) < \frac{\lambda}{CpR}\big(\frac{g(th(p + \frac{1}{t}))}{\lambda} + 1\big) e^{-b(th(p+\frac{1}{t})-R)} \to 0 \quad \text{as } b \to \infty \tag{16}$$

and thus T_L is small in probability on the scale $\frac{m}{\rho}$. The last excursion is incomplete and consists of two independent parts, an exponentially distributed holding time at the barrier and the time T_L. Hence

$$T_R - T_{N+1} \stackrel{d}{=} T_L + E,$$

where E is exponentially distributed with expected value $\frac{1}{\lambda}$ and independent of T_L. By Markov's inequality

$$P\big(\frac{\rho E}{m} > \varepsilon\big) \leq \frac{\rho}{\varepsilon m \lambda} \sim \frac{CpR}{\varepsilon m \lambda^2} e^{-Rb} \to 0 \quad \text{as } b \to \infty. \tag{17}$$

Hence, by (16) and (17), the time $\{T_R - T_{N+1}\}$ is small in probability compared with the random sum $\sum_{k=1}^{N}(T_{k+1} - T_k)$, which is of order $\frac{m}{\rho}$. We have shown that

$$\begin{aligned}
P_{b-x}\big(T_R > t\big) &= P_{b-x}\big(T_R > t \mid \{T_R < T_1\}\big) P_{b-x}\big(T_R < T_1\big) \\
&\quad + P_{b-x}\big(T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}) > t \mid \{T_1 < T_R\}\big) P_{b-x}\big(T_1 < T_R\big) \\
&\quad \sim P_{b-x}\big(T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}) > t \mid \{T_1 < T_R\}\big) \\
&\quad \sim P_{b-x}\big(\sum_{k=1}^{N}(T_{k+1} - T_k) > t \mid \{T_1 < T_R\}\big) \sim e^{-\frac{t\rho}{m}} \quad \text{as } b \to \infty,
\end{aligned}$$

and the proof of the theorem is complete.

3 An initial surplus much lower than a high dividend barrier

The risk process to be considered in this section is the same as before with the exception that the initial surplus is $u = x$ instead of $u = b - x$. In other words, suppose that

$$X(t) = x + p \int_0^t I_{(X(s) \leq b)} ds - S_t.$$

We want to derive an asymptotic formula for

$$P_x(T_R > t) = P(T_R > t \mid X(0) = x)$$

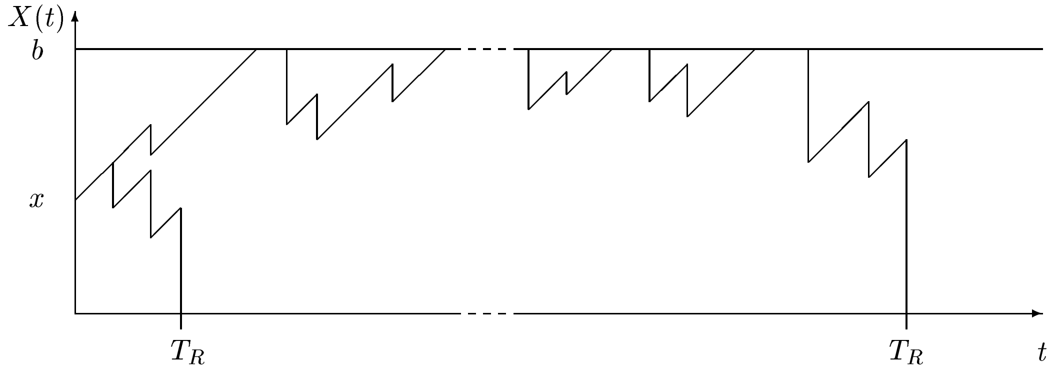


Figure 5: $X(t) = x + p\int_0^t I_{(X(s)\le b)}ds - S_t$.

when b is large and $x \ll b$. In this case, there is a non-negligible probability of ruin without visit to the dividend barrier. If the risk process visits the dividend barrier, ruin will occur after the very long exponentially distributed time in section 2, see Figure 5. We have the following theorem.

Theorem 3.1 *The time of ruin has the following asymptotic distribution*

$$\begin{cases} \frac{\rho T_R}{m} \xrightarrow{d} Exp(1) & \text{with probability } 1-\psi(x), \quad \text{as } b\to\infty, \\ T_R \xrightarrow{d} T_c \quad \text{given that } \{T_c<\infty\} & \text{with probability } \psi(x), \quad \text{as } b\to\infty, \end{cases}$$

where T_c is the time of ruin in the classical model (1) (without dividend barrier).

It is shown in Segerdahl [17] that T_c given that $\{T_c < \infty\}$ is asymptotically normal with expected value $\frac{x}{g'(R)-p} = x\bar{t}$ and variance $\frac{xg''(R)}{(g'(R)-p)^3}$ as $x \to \infty$. The proof of the theorem will follow from the lemmas below.

Lemma 3.1 *The probability of ruin without visit to the dividend barrier satisfies*

$$\psi(x,b) = \frac{\psi(x)-\psi(b)}{1-\psi(b)} \sim \psi(x) \quad \text{as } b\to\infty,$$

where $\psi(x)$ is the ruin probability in the classical model and given by (2).

Proof. Lemma (2.1) together with the *Cramér-Lundberg approximation*, $\psi(b) \sim Ce^{-Rb}$ as $b \to \infty$, yields the desired result. □

The next lemma gives a relation between the time of ruin without visit to the dividend barrier and the time of ruin in the classical model.

Lemma 3.2 *Conditional on ruin occurring before visit to the dividend barrier, the time of ruin has asymptotically the same distribution as the time of ruin T_c in the classical model given that $\{T_c < \infty\}$. In other words,*

$$P_x\big(T_R \le t \mid \{T_R < T_1\}\big) \sim P_x\big(T_c \le t \mid T_c < \infty\big) \quad \textit{as } b \to \infty.$$

Proof. Let T_b be the first passage time to the level b in the classical model. Then

$$\begin{aligned} E_x\big[e^{-sT_c}\big] &= E_x[e^{-sT_c} I_{\{T_c<T_b\}}] + E_x\big[e^{-sT_c} I_{\{T_b<T_c\}}\big] \\ &= E_x\big[e^{-sT_R} I_{\{T_R<T_1\}}\big] + E_x\big[e^{-sT_c} I_{\{T_b<T_c\}}\big], \end{aligned}$$

where in the last term $T_c = T_b + (T_c - T_b)$ and $X(T_b) = b$ on $\{T_b < T_c\}$. Moreover, the random variables T_b and $\{T_c - T_b\}$ are independent on $\{T_b < T_c\}$. By the strong Markov property the distribution of $\{T_c - T_b\}$ is the same as the conditional distribution of T_c given that $X(0) = b$. Hence

$$\begin{aligned} E_x\big[e^{-sT_c}\big] &= E_x\big[e^{-sT_R} I_{\{T_R<T_1\}}\big] + E_x\big[e^{-s(T_b+(T_c-T_b))} I_{\{T_b<T_c\}}\big] \\ &= E_x\big[e^{-sT_R} I_{\{T_R<T_1\}}\big] + E_x\big[e^{-sT_1} I_{\{T_1<T_R\}}\big] E_b[e^{-sT_c}]. \end{aligned}$$

Equation (13) with the root $z_0(s)$ and $X(0) = x$ yields

$$E_x\big[e^{-sT_1} I_{\{T_1<T_R\}}\big] \le e^{z_0(s)(b-x)},$$

where $z_0(s) < 0$ for $s > 0$. Therefore, for $s > 0$,

$$E_x\big[e^{-sT_1} I_{\{T_1<T_R\}}\big] E_b\big[e^{-sT_c}\big] \le e^{z_0(s)(b-x)} \to 0 \quad \text{as } b \to \infty,$$

and hence

$$E_x\big[e^{-sT_c}\big] \sim E_x\big[e^{-sT_R} I_{\{T_R<T_1\}}\big] \quad \text{as } b \to \infty.$$

An application of Lemma 3.1 completes the proof. □

We have shown that

$$\begin{aligned} P_x\big(T_R > t\big) \sim P_x\big(T_R > t \mid \{T_1 < T_R\}\big)\big(1 - \psi(x)\big) \\ + P_x\big(T_c > t \mid T_c < \infty\big)\psi(x) \quad \text{as } b \to \infty. \end{aligned}$$

Conditional on the risk process reaching the dividend barrier, we have

$$T_R = T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1})$$

and we know, from section 2, that $\{T_R - T_{N+1}\}$ is small in probability compared with $\sum_{k=1}^{N}(T_{k+1} - T_k)$. The next lemma gives an estimate of T_1.

Lemma 3.3 *Conditional on the risk process reaching the dividend barrier, the time to the first visit has the following bound. If* $t > \frac{1}{p-g'(0)}$ *then*

$$P_x(T_1 > (b-x)t \mid \{T_1 < T_R\}) \leq \frac{e^{-(b-x)th(p-\frac{1}{t})}}{1-\psi(x,b)}$$

and $h(p-\frac{1}{t}) > 0$.

Proof. Change the initial value from $b-x$ to x in the proof of Lemma 2.6. □

It follows, in the same way as in the conclusion of Lemma 2.6 and Lemma 2.7, that the time to the first visit is small in probability compared with the sum $\sum_{k=1}^{N}(T_{k+1}-T_k)$. Then according to section 2,

$$P_x\big(T_R > t \mid \{T_1 < T_R\}\big) \sim e^{-\frac{\rho t}{m}} \quad \text{as } b \to \infty$$

and the relevant time scale is $\frac{m}{\rho}$. The next lemma gives information about the time of ruin in the classical model.

Lemma 3.4 *Conditional on ruin occurring, the time of ruin in the classical model has the following bound. If* $t > \bar{t}$ *then*

$$P_x(T_c > xt \mid T_c < \infty) \leq \frac{1}{\psi(x)}e^{-xth(p+\frac{1}{t})}$$

and $th(p+\frac{1}{t}) > R$.

Proof. The proof is similar to the proof of Lemma 2.7, see also Martin-Löf [13]. □

We have that $P_x(\frac{\rho T_c}{m} > t \mid T_c < \infty) \approx 0$. Hence depending on whether the process reaches the dividend barrier or not, ruin occurs on different time scales. The approximation formula in Theorem 3.1 is illustrated in section 5.

4 The proportion of time the surplus is below some given level

If the risk process reaches the high dividend barrier, then the time of ruin will be very long and it is interesting to know the proportion of time the surplus is below some given level. We will use the relation to queueing theory to derive this proportion. Consider a M/G/1-queue with arrival rate λ and service times distributed according to the claim distribution F. Let V_t be the virtual waiting time at time t ($V(0) = 0$), V_t is the waiting time in queue of a hypothetical customer who arrives at time t (or the time to clear the system at time t).

Theorem 4.1 *Let $X(t)$ be the classical risk process with $p = 1$ (for simplicity) and positive safety loading. If the virtual waiting time process is defined as above then*

$$\psi(x) = P_x(\inf_{0\leq t<\infty} X(t) < 0) = P(V > x),$$

where V is the virtual waiting time in steady state.

Proof. Sketch of a proof. A rigorous proof can be found in Asmussen [4]. The key to the proof is the following coupling. Let the time T be finite. Assuming that the claims occur at the times $0 < t_1 < t_2 < \ldots < t_N < T$ and the claim amounts are $X_1, X_2, \ldots, X_N$. Let the arrivals in the M/G/1-queue be given by $0 < T - t_N < \ldots < T - t_2 < T - t_1 < T$ with corresponding service times $X_N, \ldots, X_2, X_1$, see Figure 6.

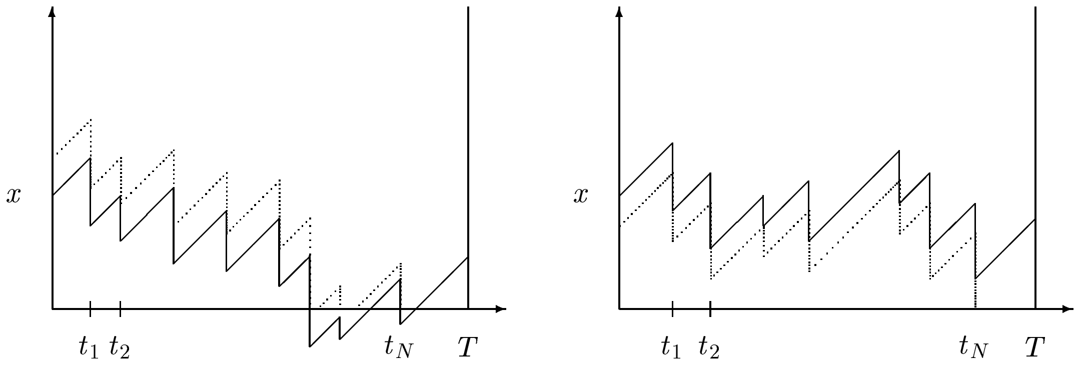


Figure 6: Dotted line V_{T-t} and solid line $X(t)$.

Figure 6 indicates that

$$\{\inf_{0\leq t\leq T} X(t) < 0\} \quad \text{iff} \quad \{V_T > x\}.$$

Hence

$$P(\inf_{0\leq t\leq T} X(t) < 0 \mid X(0) = x) = P(V_T > x).$$

Send T to infinity and the proof is complete. □

It can be shown that $V \stackrel{d}{=} W$, where W is the actual waiting time in steady state. The positive safety loading in the risk process corresponds to a traffic intensity less than 1 in the queueing process. If the traffic intensity is less than 1, then the virtual waiting time process is a regenerative process and

$$P(V > x) = \frac{E[\int_0^C I_{\{V_s > x\}} ds]}{E[C]}, \tag{18}$$

where C is the length of a cycle. The expected length is given by, see Resnick [16],

$$E[C] = \frac{1}{\lambda} + \frac{\mu}{p - \lambda\mu}.$$

Reflect the virtual waiting time process about the dividend barrier at level b. In other words, consider the process

$$R_t = b - V_t,$$

where b is the level of the dividend barrier, see Figure 7.

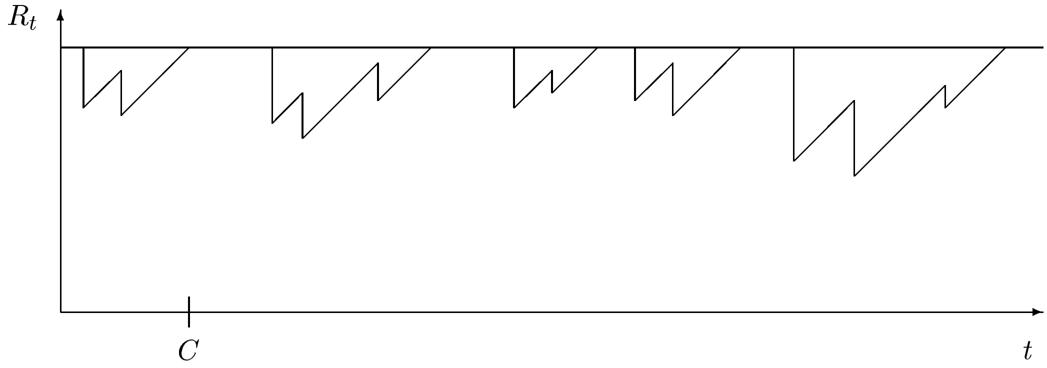


Figure 7: $R_t = b - V_t$.

This process behaves in almost the same way as the risk process at the dividend barrier. The main differences are that the risk process has a slope in between jumps not equal to 1 and it is terminated at ruin. The problem with unequal slope in between jumps is easily overcome by a time transformation. The process $X(\frac{t}{p})$ has a slope in between jumps equal to 1 and the same safety loading as the original risk process $X(t)$. The problem with termination is considered in the proof of the following theorem.

Theorem 4.2 *Conditional on the risk process reaching the dividend barrier, the proportion of time the surplus is less than $b - x$ satisfies for fixed x*

$$\begin{cases} \frac{T_{(<(b-x))}}{T_R} \xrightarrow{p} \psi(x), & x > 0, \qquad \textit{as } b \to \infty, \\ \frac{T_{(<b)}}{T_R} \xrightarrow{p} \frac{\lambda\mu}{p}, & x = 0, \qquad \textit{as } b \to \infty, \end{cases}$$

where $T_{(<(b-x))} = \int_0^{T_R} I_{\{X(s)<(b-x)\}} ds$ *(the amount of time the risk process is less than* $b - x$*).*

Proof. Conditional on the risk process reaching the dividend barrier, we can write

$$T_{(<(b-x))} = \int_0^{T_1} I_{\{X(s)<(b-x)\}} ds + \sum_{k=1}^{N} \int_{T_k}^{T_{k+1}} I_{\{X(s)<(b-x)\}} ds + \int_{T_{N+1}}^{T_R} I_{\{X(s)<(b-x)\}} ds$$

and

$$T_R = T_1 + \sum_{k=1}^{N} (T_{k+1} - T_k) + (T_R - T_{N+1}).$$

The times T_1 and $\{T_R - T_{N+1}\}$ are small in probability compared with the very long time $\sum_{k=1}^{N}(T_{k+1} - T_k)$ and in the same way, since the number of excursions will be very large, $\int_0^{T_1} I_{\{X(s)<(b-x)\}} ds$ and $\int_{T_{N+1}}^{T_R} I_{\{X(s)<(b-x)\}} ds$ are small in probability compared with $\sum_{k=1}^{N} \int_{T_k}^{T_{k+1}} I_{\{X(s)<(b-x)\}} ds$. Note that $\int_0^{T_1} I_{\{X(s)<(b-x)\}} ds \le T_1$ and $\int_{T_{N+1}}^{T_R} I_{\{X(s)<(b-x)\}} ds \le \{T_R - T_{N+1}\}$. Divide by N and the law of large numbers yields

$$\frac{T_{(<(b-x))}}{T_R} \xrightarrow{p} \frac{E[\int_{T_k}^{T_{k+1}} I_{\{X(s)<(b-x)\}} ds \mid T_{k+1} < \infty]}{E[T_{k+1} - T_k \mid T_{k+1} < \infty]} \quad \text{as } b \to \infty. \tag{19}$$

This ratio is unchanged for the time transformed risk process $X(\frac{t}{p})$. The denominator in (19) is, by Lemma 2.4, asymptotically equal to the denominator in (18). Then since, by Lemma 2.2, the probability of ruin during each excursion is very small, it follows that the numerator in (19) is asymptotically equal to the numerator in (18) and thus

$$\frac{T_{(<(b-x))}}{T_R} \xrightarrow{p} P(V > x) = \psi(x), \quad 0 < x < b, \quad \text{as } b \to \infty.$$

The last part of the theorem follows from the fact that the expected holding time at the barrier is $\frac{1}{\lambda}$ and the expected length of an excursion is asymptotically $\frac{1}{\lambda} + \frac{\mu}{p - \lambda\mu}$. The proof of the theorem is complete. □

5 A numerical example with exponentially distributed claims

The following example illustrates the approximation formula in Theorem 3.1. Suppose that the claims are exponentially distributed with expected value 1 and that the Poisson parameter is 1. Then the safety loading is $p-1$, which is assumed to be positive. Ruin occurs on different time scales depending on whether the risk process visits the dividend barrier or not. Hence it is interesting to know the Laplace transform of the time of ruin conditionally on whether the risk process visits the dividend barrier or not. Formula (13) yields

$$\begin{aligned}1 &= E_x\left[e^{z_{0,1}(s)(x-X(T_f))-sT_f}\right]\\ &= e^{z_{0,1}(s)(x-b)}E_x\left[e^{-sT_f}I_{\{T_1<T_R\}}\right] + e^{z_{0,1}(s)x}E_x\left[e^{-z_{0,1}(s)X(T_f)-sT_f}I_{\{T_R<T_1\}}\right].\end{aligned}$$

The exponentially distributed claims implies that the first passage time T_f and the deficit $X(T_f)$ are independent on the set $\{T_R < T_1\}$, see Prabhu [14]. By the forgetfulness property, the deficit is exponentially distributed with expected value 1 (note $X(T_f) < 0$ on $\{T_R < T_1\}$). Hence equation (13) yields

$$\begin{aligned}1 &= E_x[e^{z_{0,1}(s)(x-X(T_f))-sT_f}]\\ &= e^{z_{0,1}(s)(x-b)}E_x[e^{-sT_1}I_{\{T_1<T_R\}}] + \frac{e^{z_{0,1}(s)x}}{1-z_{0,1}(s)}E_x[e^{-sT_R}I_{\{T_R<T_1\}}].\end{aligned}$$

Solving these two equation yields

$$E_x[e^{-sT_1}I_{\{T_1<T_R\}}] = \frac{(1-z_0(s))e^{z_0(s)(b-x)} - (1-z_1(s))e^{z_0(s)b-z_1(s)x}}{1-z_0(s)-(1-z_1(s))e^{b(z_0(s)-z_1(s))}}, \tag{20}$$

$$E_x[e^{-sT_R}I_{\{T_R<T_1\}}] = \frac{(1-z_0(s))(1-z_1(s))e^{-xz_1(s)}(1-e^{(b-x)(z_0(s)-z_1(s))})}{1-z_0(s)-(1-z_1(s))e^{b(z_0(s)-z_1(s))}}, \tag{21}$$

where $z_{0,1}(s)$ are the roots of equation (12)

$$z_{0,1}(s) = -\frac{1}{2p}(1+s-p)\pm\frac{1}{2p}\sqrt{(1+s-p)^2+4ps}.$$

The *Cramér-Lundberg approximation* is exact in the case when the claims are exponentially distributed and the probability of ruin, without dividend barrier, is given by

$$\psi(x) = \frac{1}{p}e^{-\frac{(p-1)}{p}x}. \tag{22}$$

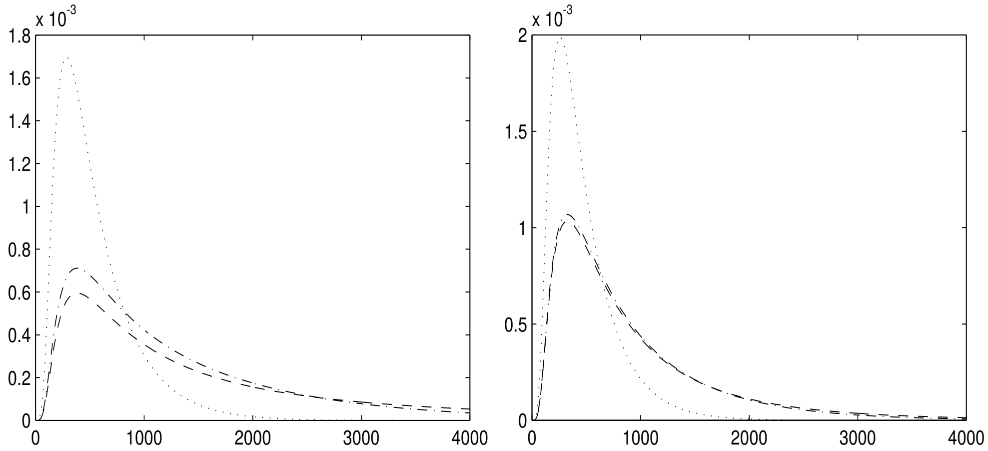
Then, by Lemma 2.2, the probability of ruin without visit to the dividend barrier is given by

$$\psi(x,b) = \frac{e^{-\frac{p-1}{p}x} - e^{-\frac{p-1}{p}b}}{p-e^{-\frac{p-1}{p}b}} \tag{23}$$

and the probability of ruin during an excursion is obtained by integration with respect to the claim distribution

$$\rho = \int_0^\infty \psi(b-x,b)F(dx) = \frac{(p-1)e^{-b\frac{(p-1)}{p}}}{p-e^{-b\frac{(p-1)}{p}}} \sim \frac{(p-1)}{p}e^{-\frac{(p-1)}{p}b} \quad \text{as } b \to \infty. \tag{24}$$

Formula (21) together with (23) yields the Laplace transform of the time of ruin given that ruin occurs without visiting the dividend barrier. The density can be obtained by numerical inversion of the Laplace transform. The numerical method used is a variant of the Fourier-series method and has been proposed by Abate, Whitt and Hill [1]. The density is plotted, for different values on the premium income rate p and the level b of the dividend barrier, in Figure 8. The values of ρ and $\psi(x,b)$



a: $p = 1.02$. b: $p = 1.05$.

Figure 8: The density of T_R given that $\{T_R < T_1\}$. Initial surplus 50. Dotted line barrier at 60, dash-dot line barrier at 120 and dashed line barrier at 240.

are given in Table 1. The Laplace transform of the time of ruin given that the risk process visits the dividend barrier can be derived by using the representation

$$T_R = T_1 + \sum_{k=1}^{N}(T_{k+1} - T_k) + (T_R - T_{N+1}),$$

$p = 1.02$		
b	$\psi(x,b)$	ρ
60	0.0939	0.0089
120	0.3028	0.0021
240	0.3622	$1.8248\cdot 10^{-4}$

$p = 1.05$		
b	$\psi(x,b)$	ρ
60	0.0353	0.0031
120	0.0852	$1.6547\cdot 10^{-4}$
240	0.0881	$5.4401\cdot 10^{-7}$

Table 1: $\psi(x,b)$ and ρ.

where the three parts are independent. The following Laplace transforms are easily calculated from (20) and (21):

$$E_x[e^{-sT_1} \mid \{T_1 < T_R\}] = \frac{(1-z_0(s))e^{z_0(s)(b-x)} - (1-z_1(s))e^{z_0(s)b - z_1(s)x}}{\big(1 - z_0(s) - (1-z_1(s))e^{b(z_0(s)-z_1(s))}\big)(1-\psi(x,b))}, \quad (25)$$

$$E[e^{-s(T_{k+1}-T_k)} \mid T_{k+1} < \infty] = \frac{1 - e^{b(z_0(s)-z_1(s))}}{\big(1 - z_0(s) - (1-z_1(s))e^{b(z_0(s)-z_1(s))}\big)(1-\rho)(1+s)}, \quad (26)$$

$$E[e^{-s(T_R - T_{N+1})}] = \frac{(z_1(s) - z_0(s))e^{-bz_1(s)}}{\big(1 - z_0(s) - (1-z_1(s))e^{b(z_0(s)-z_1(s))}\big)\rho(1+s)}. \quad (27)$$

Since the random variable N has a geometric distribution, it follows, by independence, that

$$E[e^{-s\sum_{k=1}^{N}(T_{k+1}-T_k)}] = \frac{\rho}{1 - (1-\rho)E[e^{-s(T_{k+1}-T_k)} \mid T_{k+1} < \infty]}. \quad (28)$$

Multiplying (25), (27) and (28) yields

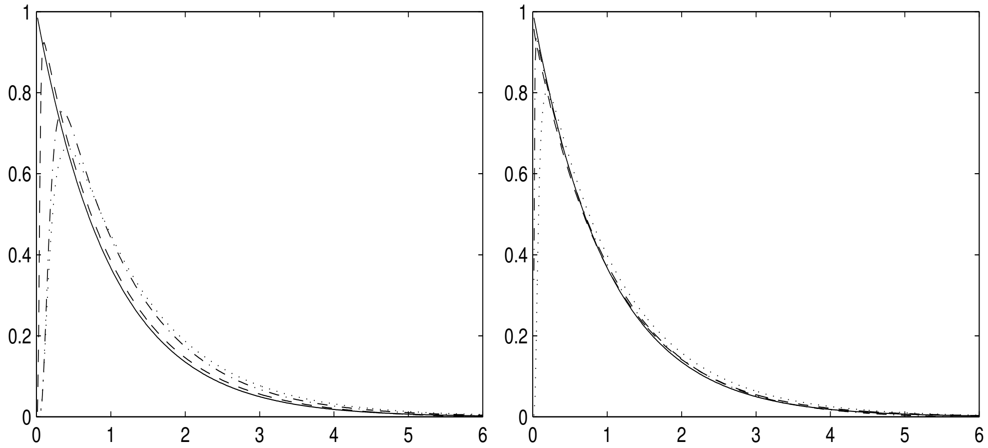
$$\begin{aligned} &E_x[e^{-sT_R} \mid \{T_1 < T_R\}] \\ &= \frac{\alpha\big((1-z_0(s))e^{(-b\alpha - xz_0(s))} - (1-z_1(s))e^{(-b\alpha - xz_1(s))}\big)\big(p - e^{-b\frac{(p-1)}{p}}\big)}{\beta\big(z_1(s)e^{-b\alpha} - z_0(s) + s\beta\big)\big(p - e^{-x\frac{(p-1)}{p}}\big)}, \end{aligned} \quad (29)$$

where

$$\begin{aligned} \alpha &= z_1(s) - z_0(s), \\ \beta &= 1 - z_0(s) - (1 - z_1(s))e^{-b\alpha} \\ z_{0,1}(s) &= -\frac{1}{2p}(1+s-p) \pm \frac{1}{2p}\sqrt{(1+s-p)^2 + 4ps}. \end{aligned}$$

The conditional density of the time of ruin, scaled by $\frac{\rho}{m}$, is plotted in Figure 9. The expected length of a complete excursion can be obtained by differentiation, of (26), which yields

$$m = 1 + \frac{p^2 - (p-1)(bp+b+p)e^{-\frac{b(p-1)}{p}} - pe^{-\frac{2b(p-1)}{p}}}{p(p-1)(p-e^{-\frac{b(p-1)}{p}})(1-e^{-\frac{b(p-1)}{p}})} \sim \frac{p}{p-1} \quad \text{as } b \to \infty.$$



a: $p = 1.02$. b: $p = 1.05$.

Figure 9: The density of $\frac{\rho T_R}{m}$ given that $\{T_1 < T_R\}$. Initial surplus 50. Dotted line barrier at 60, dash-dot line barrier at 120 and dashed line barrier at 240. Solid line exponential distribution with expected value 1.

Note the differences between the densities in Figure 9. The density of $\frac{\rho T_R}{m}$, given that $\{T_1 < T_R\}$, is the convolution of T_1, $\{T_R - T_{N+1}\}$ and $\sum_{k=1}^{N}(T_{k+1} - T_k)$. The random sum $\sum_{k=1}^{N}(T_{k+1} - T_k)$ is asymptotically exponentially distributed and the times T_1 and $\{T_R - T_{N+1}\}$ get smaller and smaller in probability compared with this sum as $b \to \infty$.

6 Discussion

An unrealistic feature of the classical model is that the only way to avoid ruin is that the surplus increases to infinity. De Finetti (1957) introduced a model with a dividend barrier which prevents this behavior. Borch developed the model and noticed that the ruin probability is 1 and concluded that the results from the classical model become irrelevant (references see [5]). In view of Theorem 3.1, either the risk process visits the high barrier and ruin occurs after a very long time $\frac{m}{\rho}$ or ruin occurs according to the classical model in a relative short time. Hence the classical model is relevant, with respect to this criticism, for reasonable time horizons and a high dividend barrier. Asymptotically the time of ruin in the classical model, if it occurs, satisfies $T_c \approx x\bar{t}$, when the initial surplus x is large. Hence if $x\bar{t} < T \ll \frac{m}{\rho}$ then the probability of ruin within the time horizon T will be given by the *Cramér-Lundberg approximation* $\psi(x) \approx Ce^{-Rx}$, when x is large and the barrier at level b is high, $x \ll b$. Note that the probability of ruin in infinite time is equal to 1.

In section 5 the Laplace transform of the time of ruin was easily calculated, in the case with exponentially distributed claims, by the use of the relation

$$1 = E_{b-x}\left[e^{z_{0,1}(s)(b-x-X(T_f))-sT_f}\right].$$

The derived Laplace transform (29) looks very complicated and it is hard to see any structure, but our analysis shows the time of ruin given that the risk process visits the barrier has asymptotically an exponential distribution. Segerdahl [18] derived the corresponding characteristic function by the use of integral-differential equations.

The same idea as in the proof of the approximation formula should work for more general premium income functions and lead to an exponentially distributed time of ruin. For example premium income functions of the following kind $p(x) = f(\frac{x}{b})$, where

$$f(x) = \begin{cases} > \lambda\mu & \text{if } x < 1 \\ < \lambda\mu & \text{if } x \leq 1 \end{cases}.$$

The risk process fluctuates about the level b and ruin will occur after a very long exponentially distributed time due to a very unlikely fluctuation. The problem is the calculation of the parameter in this exponential distribution.

Another generalization could be to consider claim distributions with heavier tails. For example the family of subexponential distributions. These distributions violates the *Cramér-Lundberg condition* and the moment generating function does not exit. A similar analysis should be possible but we can not use *Wald's fundamental identity.*

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