ON THE NUMBER OF LOST CUSTOMERS IN STATIONARY LOSS SYSTEMS IN THE LIGHT TRAFFIC CASE

TORKEL ERHARDSSON

KTH, Stockholm

Abstract. For the stationary loss systems $M/M/m/K$ and $GI/M/m/K$, we study two quantities: the number of lost customers during the time interval $(0, t]$ (the first system only), and the number of lost customers among the first n customers to arrive (both systems). We derive explicit bounds for the total variation distances between the distributions of these quantities and compound Poisson-geometric distributions. The bounds are small in the light traffic case, i.e., when the loss of a customer is a "rare" event. To prove our results, we show that the studied quantities can be interpreted as accumulated rewards of stationary renewal reward processes, embedded into the queue length process or the process of queue lengths immediately before arrivals of new customers, and apply general results by Erhardsson on compound Poisson approximation for renewal reward processes.

1. Introduction

In this paper we consider the two standard queueing systems $M/M/m/K$ and $GI/M/m/K$. In both of these, customers arrive to a system according to a renewal process in continuous time; service times for different customers are exponentially distributed and independent of each other and of the arrival process; the number of servers is m; and all customers arriving to the system at a time when K customers are already receiving service or waiting to receive service, are lost. In the first system the arrival process is homogenous Poisson. Both are examples of *loss* (or *finite waiting room*) systems.

For such systems it is clearly of great interest to know, at least under stationarity, the probability p_{loss} that an arriving customer is lost. A classical result, first derived in [5] (one of Erlang's pioneering papers which mark the beginning of queueing theory as a research area in its own right) is the *Erlang loss formula*: for the stationary loss system $M/M/K/K$ with birth intensity β and death intensity δ,

$$p_{\text{loss}} = \Big(\sum_{j=0}^{K} \frac{1}{j!}(\frac{\beta}{\delta})^j\Big)^{-1} \frac{1}{K!}(\frac{\beta}{\delta})^K.$$

Expressions for p_{loss} have subsequently been derived for more general stationary loss systems. For example, in [7] and [10] this was done for the $GI/M/K/K$ system,

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while it was shown in [9] that the Erlang loss formula holds unchanged for the $M/G/K/K$ system, i.e., for arbitrary service time distributions.

Another closely related and very important object of study for a loss system is the process of *overflows*, i.e., the point process defined as the collection of time points when an arriving customer is lost. For the $GI/M/m/K$ system, the process of overflows is a renewal process, as was first observed for the $GI/M/K/K$ system in [7]. The Laplace-Stieltjes transform of the distribution of the time between successive overflows for the $GI/M/1/K$ system was derived in [3], and this result is easily generalized to the $GI/M/m/K$ system, as pointed out in [6].

The present paper is devoted to two random variables connected to the process of overflows: Φ_t and Ψ_n, defined as the number of lost customers during the time interval $(0, t]$, and the number of lost customers among the first n customers to arrive, respectively. These have previously received comparatively little attention. Of course, Φ_t is just the number of overflows during $(0, t]$, so for the $GI/M/m/K$ system it follows from the above mentioned facts that the distribution of Φ_t is, in principle, known. However, in practice it is too cumbersome to compute this distribution exactly (or numerically) unless K and t are small. There also exists a central limit theorem for the number of renewals during an interval $(0, t]$, which can be applied to Φ_t (see Section VI.4 in [1]), but this theorem is of no use when the loss of a customer is a "rare" event (i.e., in the light traffic case).

Here, for the stationary $M/M/m/K$ system, we instead derive an explicit bound for the total variation distance between the distribution of Φ_t and a *compound Poisson-geometric* distribution. The two parameters of this approximating distribution are explicitly calculated. The bound is small in the light traffic case. Hence, our result complements the central limit theorem, and can in the light traffic case be seen as a distributional counterpart to the Erlang loss formula. Appropriately, for the $M/M/K/K$ system the mean of our approximating distribution is $E(\Phi_t) = \beta t p_{\text{loss}}$.

For the $GI/M/m/K$ system, we derive a bound for the total variation distance between the distribution of Ψ_n and another compound Poisson-geometric distribution. Here, the total variation distance bound and the two parameters of the approximating distribution can not in general be given in a simple explicit form, but they can be calculated by solving four systems of linear equations of dimension at most K. Just as for Φ_t, in the light traffic case the bound is small, and the result can be seen as a distributional counterpart to the Erlang loss formula. For the $M/M/m/K$ system we derive completely explicit expressions for all quantities involved. For the $M/M/K/K$ system the mean of our approximating distribution is $E(\Psi_n) = n p_{\text{loss}}$.

To prove our results, we show that Φ_t can be interpreted as the accumulated reward of a stationary renewal reward process embedded into the queue length process, where the renewals occur at those time points when the queue length process has just made a transition from state $\{0\}$ to state $\{1\}$. This permits the use of general results on compound Poisson approximation for renewal reward processes given in [4]. Ψ_n is handled by a similar argument applied to the (discrete time) Markov chain of queue lengths immediately before arrivals of new customers.

The paper is organized as follows. In Section 2 we give some notational conventions and definitions, and state some well-known results about birth and death processes. In Section 3 the results about the random variable Φ_t for the stationary $M/M/m/K$ system are stated and proven. In Section 4 the results about the random variable Ψ_n for the $GI/M/m/K$ and $M/M/m/K$ systems are stated and proven. Section 5 contains some numerical evaluations of the bounds derived in the previous sections.

2. Preliminaries

We use the following notation for sets of numbers: $R =$ the real numbers, $Z =$ the integers, $R_+ = [0, \infty)$, $R'_+ = (0, \infty)$, $Z_+ = \{0, 1, 2, ...\}$, and $Z'_+ = \{1, 2, ...\}$.

For any random element X in any measurable space $(S, \mathcal{F})$, we denote the distribution of X by $\mathcal{L}(X)$.

For any finite set S and $I = R$ or $I = R_+$, we denote by $\mathcal{D}(I, S)$ the space of all functions $f : I \to S$ which are right-continuous and have left hand limits at every point. We consider $\mathcal{D}(I, S)$ as a measurable space equipped with the σ-algebra generated by the finite-dimensional sets. We define, for each $A \subset S$ and $t \in I$, the functional $\tau^t_A : \mathcal{D}(I, S) \to R'_+ \cup \{\infty\}$ by

$$\tau^t_A(f) := \inf\{u \in R'_+; f(t + u) \in A\} \qquad \forall f \in \mathcal{D}(I, S),$$

and the functional $\bar{\tau}^t_A : \mathcal{D}(I, S) \to R_+ \cup \{\infty\}$ by

$$\bar{\tau}^t_A(f) := \inf\{u \in R_+; f(t + u) \in A\} \qquad \forall f \in \mathcal{D}(I, S),$$

with the convention that $\inf \emptyset = \infty$. Similarly, for $I = Z$ or $I = Z_+$, we denote by S^I the space of all sequences (functions) $f : I \to S$. We define, for each $A \subset S$ and $t \in I$, the functional $\tau^t_A : S^I \to Z'_+ \cup \{\infty\}$ by

$$\tau^t_A(f) := \inf\{u \in Z'_+; f(t + u) \in A\} \qquad \forall f \in S^I,$$

and the functional $\bar{\tau}^t_A : S^I \to Z_+ \cup \{\infty\}$ by

$$\bar{\tau}^t_A(f) := \inf\{u \in Z_+; f(t + u) \in A\} \qquad \forall f \in S^I.$$

For brevity we will use the notation $\tau_A(\cdot) := \tau^0_A(\cdot)$ and $\bar{\tau}_A(\cdot) := \bar{\tau}^0_A(\cdot)$.

For $I = R$ or $I = Z$ we define $\mathcal{N}(I \times Z_+)$ as the space of counting measures (i.e., integer valued Radon measures) on $I \times Z_+$. We consider $\mathcal{N}(I \times Z_+)$ as a measurable space equipped with the σ-algebra generated by the finite-dimensional sets. A random element in $\mathcal{N}(I \times Z_+)$ is called a *point process*. For any random sequence $\{(X_i, Y_i); i \in Z\}$ such that (X_i, Y_i) takes values in $I \times Z_+$ for each $i \in Z$, we define the point process ξ *generated by* $\{(X_i, Y_i); i \in Z\}$ as

$$\xi(\cdot) := \sum_{i=-\infty}^{\infty} I\{(X_i, Y_i) \in \cdot\}.$$

A (Palm version of a) *renewal reward process* (in the sense of Definition 4.1 in [4]) is a point process generated by a random sequence $\{(X_i, Y_i); i \in Z\}$ such that (X_i, Y_i) takes values in $I \times Z_+$ for each $i \in Z$, $X_0 \equiv 0$, and $\{(X_{i+1} - X_i, Y_i); i \in Z\}$ is an I.I.D. sequence.

For any two probability measures ν_1 and ν_2 on any measurable space $(S, \mathcal{F})$ we define the *total variation distance* $d_{TV}(\nu_1, \nu_2)$ by

$$d_{TV}(\nu_1, \nu_2) := \sup_{A \in \mathcal{F}} |\nu_1(A) - \nu_2(A)|.$$

A nonnegative random variable W is said to have a *compound Poisson* distribution POIS(ν), where ν is a measure on R'_+ such that $\int_0^\infty (1 \wedge x) d\nu(x) < \infty$, if the characteristic function of W is $E(e^{itW}) = \exp(-\int_{R'_+} (1 - e^{itx}) d\nu(x)) \quad \forall t \in R$. If

$\lambda := \nu(R'_+) < \infty$, then $\mathrm{POIS}(\nu) = \mathcal{L}(\sum_{i=1}^{U} T_i)$, where the variables $\{T_i; i \in Z'_+\}$ and U are independent, $\mathcal{L}(T_i) = \nu/\lambda \ \ \forall i \in Z'_+$, and $U \sim \mathrm{Po}(\lambda)$. If $\nu = \lambda\nu_a$ for some $a > 0$, where $\nu_a(k) := (1-a)^{k-1}a \ \ \forall k \in Z'_+$, then $\mathrm{POIS}(\nu)$ is called a *compound Poisson-geometric* (or *Pólya-Aeppli*) distribution.

By a *birth and death* process η on the state space $\{0,...,M\}$ ($M \in Z'_+$), we mean a Markovian pure jump process (for a definition of such a process, see Chapter II in [1]) on $\{0,...,M\}$, with an intensity matrix Q whose off-diagonal elements are given by

$$Q_{i,j} := \begin{cases} \beta_i, & \text{if } i \leq M-1 \text{ and } j = i+1; \\ \delta_i, & \text{if } i \geq 1 \text{ and } j = i-1; \\ 0, & \text{otherwise.} \end{cases}$$

Define, for convenience, for each $i, k \in \{0,...,M\}$ such that $i \leq k$,

$$\pi_{i,k} := \begin{cases} (\beta_i/\delta_k) \prod_{j=i+1}^{k-1} (\beta_j/\delta_j), & \text{if } i < k; \\ 1, & \text{if } i = k. \end{cases}$$

It is well-known that η has a unique stationary distribution μ, defined by

$$\mu(k) = \frac{\pi_{0,k}}{\sum_{i=0}^{M} \pi_{0,i}} \qquad \forall k \in \{0,...,M\}. \tag{2.1}$$

It is also well-known that, for each $a, k, b \in \{0,...,M\}$ such that $a < k < b$,

$$P(\tau_a(\eta) < \tau_b(\eta) | \eta_0 = k) = (\sum_{j=a}^{b-1} \frac{1}{\beta_j \pi_{0,j}})^{-1} \sum_{i=k}^{b-1} \frac{1}{\beta_i \pi_{0,i}}, \tag{2.2}$$

and that, for each $a, k \in \{0,...,M\}$ such that $a < k$,

$$E(\tau_a(\eta) | \eta_0 = k) = \sum_{i=a}^{k-1} \sum_{j=i+1}^{M} \frac{\pi_{i,j}}{\beta_i}. \tag{2.3}$$

By a *birth-death* chain η on the state space $\{0,...,M\}$, we mean an irreducible Markov chain on the state space $\{0,...,M\}$ with a transition matrix p given by

$$p_{i,j} := \begin{cases} p_i, & \text{if } i \leq M-1 \text{ and } j = i+1; \\ q_i, & \text{if } i \geq 1 \text{ and } j = i-1; \\ r_i, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

The results (2.1), (2.1) and (2.3) hold for the birth-death chain η as well, if β_i and δ_i are replaced by p_i and q_i for each $i \in \{0,...,M\}$.

3. The number of lost customers during $(0, t]$

Theorem 3.1. *Let Φ_t be the number of lost customers during the time $(0,t]$ in a stationary $M/M/m/K$ queueing system ($m \leq K$) with arrival intensity β and service intensity δ. Let $\sigma_0 := 1$ and $\sigma_j := \prod_{i=1}^{j}(i \wedge m) \ \ \forall j \in \{1,...,K\}$. Then,*

$$d_{TV}(\mathcal{L}(\Phi_t), \mathrm{POIS}(\lambda\nu_\theta)) \leq H(\lambda,\theta) 3\beta t (\sum_{j=0}^{K} \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-2} \frac{1}{\sigma_K^2}(\frac{\beta}{\delta})^{2K}$$

$$\times\Big(2\sum_{i=0}^{K-2}\sum_{j=i+1}^{K}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}+3+\frac{2\beta}{m\delta}+2(\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\sum_{r=1}^{K}\frac{1}{\sigma_r}(\frac{\beta}{\delta})^r\sum_{i=0}^{r-1}\sum_{j=i+1}^{K}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}\Big)$$

$$+2(\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\Big((\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1}\sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}$$

$$+\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1}+(1+\frac{\beta}{m\delta})^{-1}(\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1}\sum_{i=0}^{K-1}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i}\Big),$$

where $\nu_\theta(k) := (1-\theta)^{k-1}\theta \quad \forall k \in Z'_+$, *and:*

$$\lambda=\beta t(\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K(\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1};\quad \theta=(\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1};$$

$$H(\lambda,\theta)\le\begin{cases}(\frac{1}{\lambda\theta}\wedge 1)\exp(\lambda), & \text{if } \theta\in(0,1);\\ \frac{1}{\lambda\theta(2\theta-1)}\Big(\frac{1}{4\lambda\theta(2\theta-1)}+\log^+(2\lambda\theta(2\theta-1))\Big)\wedge 1, & \text{if } \theta\in[\frac{1}{2},1);\\ \frac{\theta^2}{\lambda(5\theta-4)}, & \text{if } \theta\in(\frac{4}{5},1).\end{cases}$$

Proof. Consider an $M/M/m/K$ queueing system to which the first customer arrives at time $t = 0$. Define the random element η^+ in $\mathcal{D}(R_+, \{0, ..., K+2\})$ as the number of customers in the system at each time $t \in R_+$, except if the buffer is full, in which case η^+_t also gives information about how many customers have been lost since the buffer became full: $\eta^+_t = \{K\}$ if 0 customers have been lost, $\eta^+_t = \{K + 1\}$ if an odd number of customers have been lost, and $\eta^+_t = \{K + 2\}$ if an even number ≥ 2 have been lost. η^+ is a Markovian pure jump process on $\{0, ..., K + 2\}$ with an intensity matrix Q whose off-diagonal elements are given by

$$Q_{i,j}:=\begin{cases}\beta, & \text{if } i\le K+1 \text{ and } j=i+1;\\ \beta, & \text{if } i=K+2 \text{ and } j=K+1;\\ (i\wedge m)\delta, & \text{if } 1\le i\le K \text{ and } j=i-1;\\ m\delta, & \text{if } i\ge K+1 \text{ and } j=K-1;\\ 0, & \text{otherwise.}\end{cases}$$

This can be proven by first showing that the random sequence $\{(\eta^+_{S_i}, S_i - S_{i-1}); i \in Z'_+\}$, where $0 = S_0 < S_1 < S_2 < ...$ are the jump times of η^+, is a Markov chain on the state space $\{0, ..., K + 2\} \times R'_+$ with transition function p defined by

$$p((i,x),\{j\}\times(y,\infty)):=\frac{Q_{i,j}}{|Q_{i,i}|}\exp(-|Q_{i,i}|y)\qquad \forall i,j\in\{0,...,K+2\}, x,y\in R'_+.$$

The strong Markov property implies that η^+ is *regenerative* with regeneration times $\{0\} \cup \{t \in R'_+; \eta^+_{t-} = 0, \eta^+_t = 1\}$. Since $E(\tau_0(\eta^+)) < \infty$, there exists a random element η in $\mathcal{D}(R, \{0, ..., K + 2\})$ which is a stationary version of η^+ (with index set R). Clearly,

$$\Phi_t = \text{card}\{s \in (0, t]; \eta_{s-} \geq K + 1, \eta_s \neq \eta_{s-}\} \qquad \forall t \in R_+.$$

Define the random sequence $\{(X_i, Y_i); i \in Z\}$ as follows. Let $\{X_i; i \in Z\}$ (where $... < X_{-1} < X_0 \leq 0 < X_1 < ...$) be the random times $\{t \in R; \eta_{t-} = 0, \eta_t = 1\}$, and let

$$Y_i := \text{card}\{t \in R; \eta_{t-} \geq K+1, \eta_t \neq \eta_{t-}, X_i \leq t < X_{i+1}\} \qquad \forall i \in Z.$$

Define the random element ξ in $\mathcal{N}(R, Z_+)$ as the point process generated by $\{(X_i, Y_i); i \in Z\}$. Define the random element $(\eta^{\text{o}}, \xi^{\text{o}})$ in $\mathcal{D}(R, \{0, ..., K+2\}) \times \mathcal{N}(R \times Z_+)$ as a Palm version of (η, ξ) (see Section 3.2 in [8]), and define the random sequence $\{(X_i^{\text{o}}, Y_i^{\text{o}}); i \in Z\}$ as the coordinates of the points of ξ^{o}. It is clear that $\mathcal{L}(\eta_t^{\text{o}}; t \in R_+) = \mathcal{L}(\eta^+; t \in R_+)$. Define also $\{T_i^{\text{o}}; i \in Z\}$ by $T_i^{\text{o}} := X_{i+1}^{\text{o}} - X_i^{\text{o}} \ \ \forall i \in Z$. The relation between $(\eta^{\text{o}}, \xi^{\text{o}})$ and (η, ξ) is given by the Palm inversion formula: for each measurable function $g : \mathcal{D}(R, \{0, ..., K+2\}) \times \mathcal{N}(R \times Z_+) \to R_+$, it holds that

$$E(g(\eta, \xi)) = \frac{E\big(\int_0^{T_0^{\text{o}}} g(\theta_t(\eta^{\text{o}}, \xi^{\text{o}}))dt\big)}{E(T_0^{\text{o}})}, \tag{3.1}$$

where $\theta : R \times \mathcal{D}(R, \{0, ..., K+2\}) \times \mathcal{N}(R \times Z_+) \to \mathcal{D}(R, \{0, ..., K+2\}) \times \mathcal{N}(R \times Z_+)$ is the shift operator. With the notation $\mu_Y := \mathcal{L}(Y_0^{\text{o}})$ and $\mu_Y' := \mu_Y(\cdot \cap Z_+')$, the triangle inequality implies that

$$d_{TV}(\mathcal{L}(\Phi_t), \text{POIS}(\frac{t\mu_Y'}{E(T_0^{\text{o}})})) \leq d_{TV}(\mathcal{L}(\Phi_t), \mathcal{L}(\int_{(0,t] \times Z_+'} v d\xi(u, v)))$$

$$+ d_{TV}(\mathcal{L}(\int_{(0,t] \times Z_+'} v d\xi(u, v)), \text{POIS}(\frac{t\mu_Y'}{E(T_0^{\text{o}})})),$$

and the basic coupling inequality implies that $d_{TV}(\mathcal{L}(\Phi_t), \mathcal{L}(\int_{(0,t] \times Z_+'} v d\xi(u, v))) \leq 2P(\bar{\tau}_{\{K+1, K+2\}}(\eta) < \bar{\tau}_0(\eta))$. For the second term, we note that ξ is a stationary (version of a) renewal reward process in the sense of Definition 4.1 in [4]. Hence, Theorem 5.1 in [4] gives the bound

$$\begin{aligned} &d_{TV}(\mathcal{L}(\int_{(0,t] \times Z_+'} v d\xi(u, v)), \text{POIS}(\frac{t\mu_Y'}{E(T_0^{\text{o}})})) \\ &\leq H(\frac{t\mu_Y'}{E(T_0^{\text{o}})}) \frac{3tE(Y_0^{\text{o}})}{E(T_0^{\text{o}})} \Big(\frac{E(T_0^{\text{o}} Y_0^{\text{o}})}{E(T_0^{\text{o}})} + \frac{E((T_0^{\text{o}})^2) E(Y_0^{\text{o}})}{E(T_0^{\text{o}})^2} \Big), \end{aligned} \tag{3.2}$$

where $H(E(T_0^{\text{o}})^{-1} t\mu_Y') \leq ((E(T_0^{\text{o}})^{-1} t\mu_Y(1))^{-1} \wedge 1) \exp(E(T_0^{\text{o}})^{-1} t\mu_Y(Z_+'))$, unless $\{k\mu_Y(k); k \in Z_+'\}$ is monotonically decreasing towards 0, in which case

$$H(\frac{t\mu_Y'}{E(T_0^{\text{o}})}) \leq \frac{1}{\Delta_Y(1)} \Big(\frac{1}{4\Delta_Y(1)} + \log^+ 2\Delta_Y(1) \Big) \wedge 1,$$

where $\Delta_Y(1) := E(T_0^{\text{o}})^{-1} t(\mu_Y(1) - 2\mu_Y(2))$. Moreover, a recent result, Theorem 2.5 in [2], tells us that if

$$\kappa := \frac{\sum_{k=2}^{\infty} k(k-1)\mu_Y(k)}{\sum_{k=1}^{\infty} k\mu_Y(k)} < \frac{1}{2},$$

then it also holds that

$$H(\frac{t\mu_Y'}{E(T_0^{\text{o}})}) \leq \frac{E(T_0^{\text{o}})}{t(1-2\kappa) \sum_{k=1}^{\infty} k\mu_Y(k)}.$$

We need to compute the quantities appearing in (3.2). From (3.1) we get that $E(T_0^{\text{o}}) = E(\tau_1(\eta) | \eta_0 = 0)/\mu(0)$, where μ is the stationary distribution of η. It is well-known that $E(\tau_1(\eta)|\eta_0 = 0) = \beta^{-1}$, and it is easy to check that μ is given by

$$\begin{aligned}
\mu(r) &= \big(\sum_{j=0}^{K} \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j\big)^{-1}\frac{1}{\sigma_r}(\frac{\beta}{\delta})^r \qquad \forall r \in \{0, ..., K-1\};\\
\mu(K) &= \frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1}\mu(K-1);\\
\mu(K+1) &= (\frac{\beta}{m\delta})^2(1+\frac{2\beta}{m\delta})^{-1}\mu(K-1);\\
\mu(K+2) &= (\frac{\beta}{m\delta})^3(1+\frac{\beta}{m\delta})^{-1}(1+\frac{2\beta}{m\delta})^{-1}\mu(K-1),
\end{aligned}$$

implying that $E(T_0^{\circ}) = \beta^{-1}\sum_{j=0}^{K}\sigma_j^{-1}(\beta/\delta)^j$. Next, define the random element η^K in $\mathcal{D}(R, \{0, ..., K\})$ by $\eta_t^K := \eta_t \wedge K \ \ \forall t \in R$. It is not difficult to show that η^K is a stationary birth and death process with an intensity matrix Q^K with off-diagonal elements given by

$$Q_{i,j}^K := \begin{cases} \beta, & \text{if } i \le K-1 \text{ and } j = i+1;\\ (i\wedge m)\delta, & \text{if } i \ge 1 \text{ and } j = i-1;\\ 0, & \text{otherwise.}\end{cases}$$

Denote the stationary distribution of η^K by μ^K. From (3.1), the definition of η^K, (2.1) and (2.3), we get:

$$\frac{E((T_0^{\circ})^2)}{2E(T_0^{\circ})} = E(X_1) = \sum_{r=1}^{K} E(\tau_0(\eta^K)|\eta_0^K = r)\mu^K(r) + E(\tau_1(\eta^K)|\eta_0^K = 0)$$

$$= \frac{1}{\beta}\big(\sum_{r=1}^{K}\mu^K(r)\sum_{i=0}^{r-1}\sum_{j=i+1}^{K}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i} + 1\big)$$

$$= \frac{1}{\beta}\big((\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\sum_{r=1}^{K}\frac{1}{\sigma_r}(\frac{\beta}{\delta})^r\sum_{i=0}^{r-1}\sum_{j=i+1}^{K}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i} + 1\big).$$

We now turn our attention to μ'_Y. The strong Markov property implies that $\mu_Y(k) = P(\tau_{K+1}(\eta) < \tau_0(\eta)|\eta_0 = 1)(1-\theta)^{k-1}\theta \ \ \forall k \in Z'_+$, where $\theta = P(\tau_0(\eta) < \tau_{K+1}(\eta)|\eta_0 = K-1)m\delta/(\beta+m\delta)$. This implies that $\mathrm{POIS}(E(T_0^{\circ})^{-1}t\mu'_Y) = \mathrm{POIS}(\lambda\nu_\theta)$, where $\lambda = E(T_0^{\circ})^{-1}tP(\tau_{K+1}(\eta) < \tau_0(\eta)|\eta_0 = 1)$, and also, after a little bit of work, that $H(\lambda,\theta) := H(E(T_0^{\circ})^{-1}t\mu'_Y)$ satisfies the inequalities claimed in Theorem 3.1. Using (2.2) and the fact that η is "almost" a birth and death process, we get:

$$P(\tau_{K+1}(\eta) < \tau_0(\eta)|\eta_0 = 1) = \big(\sum_{j=0}^{K}\sigma_j(\frac{\delta}{\beta})^j\big)^{-1} = \frac{1}{\sigma_K}(\frac{\beta}{\delta})^K(\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1};$$

$$\theta = \big(\sum_{j=0}^{K}\sigma_j(\frac{\delta}{\beta})^j\big)^{-1}\sigma_{K-1}(\frac{\delta}{\beta})^{K-1}(1+\frac{m\delta}{\beta})\frac{m\delta}{\beta+m\delta} = (\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1}.$$

Hence, $E(Y_0^{\circ}) = \mu_Y(Z'_+)\theta^{-1} = \sigma_K^{-1}(\beta/\delta)^K$. Similarly,

$$P(\bar{\tau}_{\{K+1,K+2\}}(\eta) < \bar{\tau}_0(\eta)) = \mu(\{K+1,K+2\}) + \sum_{r=1}^{K}\mu(r)(\sum_{i=0}^{K}\sigma_i(\frac{\delta}{\beta})^i)^{-1}\sum_{j=0}^{r-1}\sigma_j(\frac{\delta}{\beta})^j$$

$$= (\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1} + (\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}(\sum_{i=0}^{K}\sigma_i(\frac{\delta}{\beta})^i)^{-1}$$

$$\times\Big(\sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i} + \frac{1}{\sigma_K}(\frac{\beta}{\delta})^K(1+\frac{\beta}{m\delta})^{-1}\sum_{j=0}^{K-1}\sigma_j(\frac{\delta}{\beta})^j\Big)$$

$$= (\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\Big((\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1}\sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}$$

$$+\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1} + (1+\frac{\beta}{m\delta})^{-1}(\sum_{i=0}^{K}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1}\sum_{i=0}^{K-1}\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i}\Big).$$

It remains to calculate $E(T_0^{\circ}Y_0^{\circ})$. Define the random element η^R in $\mathcal{D}(R,\{0,...,K+2\})$ by $\eta_t^R := \eta_{(-t)-}\ \ \forall t \in R$; i.e., η^R is the *reversed* Markovian pure jump process corresponding to η. It is well-known, see e.g. Section II.5 in [1], that η^R is a stationary Markovian pure jump process with intensity matrix Q^R defined by $Q^R_{i,j} := \mu(j)Q_{j,i}/\mu(i)\ \ \forall i,j \in \{0,...,K+2\}$. Also, define the random element $\eta^{K,R}$ in $\mathcal{D}(R,\{0,...,K\})$ by $\eta_t^{K,R} := \eta^K_{(-t)-}\ \forall t \in R$. η^K is *reversible* (since it is a birth and death process), so $\eta^{K,R}$ has intensity matrix $Q^{K,R} = Q^K$. Dominated convergence implies that $E(T_0^{\circ}Y_0^{\circ}) = \lim_{n\to\infty}E(T_0^{\circ}Y_0^{1/n})$, where

$$Y_0^{1/n} := n\int_0^{(T_0^{\circ}-1/n)^+} I\{\eta_t^{\circ} \geq K+1, \eta^{\circ}_{t+1/n} \neq \eta_t^{\circ}\}dt \qquad \forall n \in Z'_+,$$

so (3.1) gives

$$\frac{E(T_0^{\circ}Y_0^{\circ})}{E(T_0^{\circ})} = \lim_{n\to\infty} nE\big((X_1-X_0)I\{X_1 > \frac{1}{n}, \eta_0 \geq K+1, \eta_{1/n} \neq \eta_0\}\big).$$

We note that, on the set $\{\eta_0 \geq K+1, \eta_{1/n} \neq \eta_0\}$, only four values are possible for the pair $(\eta_0, \eta_{1/n})$, so the expectation can be written as a sum of four terms. Conditioning on $\sigma(\eta_t; t \in R_+)$ and $\sigma(\eta_t; t \in (-\infty, 1/n])$ for each of these terms gives:

$$\lim_{n\to\infty} nE\big((X_1-X_0)I\{X_1 > \frac{1}{n}, \eta_0 \geq K+1, \eta_{1/n} \neq \eta_0\}\big)$$

$$= \lim_{n\to\infty} n\mu(K+1)m\delta\frac{1}{n}\big(E(\tau_0(\eta^R)|\eta_0 = K+1) + \frac{1}{n} + E(\tau_0(\eta)|\eta_0 = K-1) + \frac{1}{\beta}\big)$$

$$+ \lim_{n\to\infty} n\mu(K+1)\beta\frac{1}{n}\big(E(\tau_0(\eta^R)|\eta_0 = K+1) + \frac{1}{n} + E(\tau_0(\eta)|\eta_0 = K+2) + \frac{1}{\beta}\big)$$

$$+ \lim_{n\to\infty} n\mu(K+2)m\delta\frac{1}{n}\big(E(\tau_0(\eta^R)|\eta_0 = K+2) + \frac{1}{n} + E(\tau_0(\eta)|\eta_0 = K-1) + \frac{1}{\beta}\big)$$

$$+\lim_{n\to\infty} n\mu(K+2)\beta\frac{1}{n}\big(E(\tau_0(\eta^R)|\eta_0=K+2)+\frac{1}{n}+E(\tau_0(\eta)|\eta_0=K+1)+\frac{1}{\beta}\big)$$

$$=\mu(K+1)m\delta\big(E(\tau_{K-1}(\eta^R)|\eta_0=K+1)+2E(\tau_0(\eta)|\eta_0=K-1)+\frac{1}{\beta}\big)$$

$$+\mu(K+1)\beta\big(E(\tau_{K-1}(\eta^R)|\eta_0=K+1)+2E(\tau_0(\eta)|\eta_0=K-1)+\frac{1}{m\delta}+\frac{1}{\beta}\big)$$

$$+\mu(K+2)m\delta\big(E(\tau_{K-1}(\eta^R)|\eta_0=K+2)+2E(\tau_0(\eta)|\eta_0=K-1)+\frac{1}{\beta}\big)$$

$$+\mu(K+2)\beta\big(E(\tau_{K-1}(\eta^R)|\eta_0=K+2)+2E(\tau_0(\eta)|\eta_0=K-1)+\frac{1}{m\delta}+\frac{1}{\beta}\big),$$

where we used the fact that

$$E(\tau_0(\eta^R)|\eta_0=K-1)=E(\tau_0(\eta^{K,R})|\eta_0^{K,R}=K-1)$$

$$=E(\tau_0(\eta^K)|\eta_0^K=K-1)=E(\tau_0(\eta)|\eta_0=K-1),$$

and also that $E(\tau_{K-1}(\eta)|\eta_0=K+1)=E(\tau_{K-1}(\eta)|\eta_0=K+2)=1/(m\delta)$. Moreover, using (2.3) it is not difficult to show that

$$E(\tau_{K-1}(\eta^R)|\eta_0=K+1)=\frac{1}{\beta}\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})(1+\frac{2\beta}{m\delta})^{-1}$$

$$+\frac{1}{\beta}(\frac{\beta}{m\delta})^3(1+\frac{\beta}{m\delta})^{-1}(1+\frac{2\beta}{m\delta})^{-1}+\frac{1}{\beta}\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1};$$

$$E(\tau_{K-1}(\eta^R)|\eta_0=K+2)=E(\tau_{K-1}(\eta^R)|\eta_0=K+1)+\frac{1}{\beta}\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1},$$

so we get:

$$\frac{E(T_0^\circ Y_0^\circ)}{E(T_0^\circ)}=\mu(\{K+1,K+2\})(\beta+m\delta)\big(E(\tau_{K-1}(\eta^R)|\eta_0=K+1)$$

$$+2E(\tau_0(\eta)|\eta_0=K-1)\big)+(1+\frac{\beta}{m\delta}+\frac{m\delta}{\beta})\mu(K+1)+(2+\frac{\beta}{m\delta}+\frac{m\delta}{\beta})\mu(K+2)$$

$$=(\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\Big(\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})(1+\frac{2\beta}{m\delta})^{-1}$$

$$+(\frac{\beta}{m\delta})^3(1+\frac{\beta}{m\delta})^{-1}(1+\frac{2\beta}{m\delta})^{-1}+\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1}+2\sum_{i=0}^{K-2}\sum_{j=i+1}^{K}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}$$

$$+(1+\frac{\beta}{m\delta}+(\frac{\beta}{m\delta})^2)(1+\frac{2\beta}{m\delta})^{-1}+(1+\frac{2\beta}{m\delta}+(\frac{\beta}{m\delta})^2)\frac{\beta}{m\delta}(1+\frac{\beta}{m\delta})^{-1}(1+\frac{2\beta}{m\delta})^{-1}\Big)$$

$$=(\sum_{j=0}^{K}\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\Big(2\sum_{i=0}^{K-2}\sum_{j=i+1}^{K}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}+1+\frac{2\beta}{m\delta}\Big).$$

□

Example 3.1. Let Φ_t be the number of lost arriving customers during the time $(0,t]$ in a stationary $M/M/1/K$ queueing system with arrival intensity β and service intensity δ. Let $\rho=\beta/\delta<1$. Then $\sigma_j=1\ \ \forall j\in\{1,...,K\}$, and

$$d_{TV}(\mathcal{L}(\Phi_t),\mathrm{POIS}(\lambda\nu_\theta))\leq H(\lambda,\theta)3\beta t\frac{(1-\rho)^2}{(1-\rho^{K+1})^2}\rho^{2K}$$

$$\times\Big(\frac{2\rho}{1-\rho}(K-1-\frac{\rho^2(1-\rho^{K-1})}{1-\rho})+3+2\rho$$

$$+\frac{2\rho^2}{(1-\rho)^2(1-\rho^{K+1})}(1-\rho^{2K+1}-(2K+1)(1-\rho)\rho^K)\Big)$$

$$+\frac{2(1-\rho)\rho^{K+1}}{1-\rho^{K+1}}\Big(\frac{1}{1-\rho^{K+1}}(K-1-\frac{\rho(1-\rho^{K-1})}{1-\rho})+\frac{1}{1+\rho}(1+\frac{1-\rho^K}{1-\rho^{K+1}})\Big),$$

where

$$\lambda=\beta t\frac{(1-\rho)^2}{(1-\rho^{K+1})^2}\rho^K;\qquad\theta=\frac{1-\rho}{1-\rho^{K+1}}.$$

In particular, if $\rho<1$ is constant and $t:=t(K)$ is chosen so that

$$\lim_{K\to\infty}E(\Phi_t)=\lim_{K\to\infty}\beta t\frac{1-\rho}{1-\rho^{K+1}}\rho^K=c>0,$$

then, as $K\to\infty$, $d_{TV}(\mathcal{L}(\Phi_t),\mathrm{POIS}(\lambda\nu_\theta))=\mathrm{O}(K\rho^K)$, and $\mathrm{POIS}(\lambda\nu_\theta)$ converges weakly to $\mathrm{POIS}(c(1-\rho)\nu_{(1-\rho)})$.

Example 3.2. Let Φ_t be the number of lost arriving customers during the time $(0,t]$ in a stationary $M/M/K/K$ queueing system with arrival intensity β and service intensity δ. Then $\sigma_j=j!\ \ \forall j\in\{1,...,K\}$. If $\rho=\beta/\delta$ is constant and $t:=t(K)$ is chosen so that

$$\lim_{K\to\infty}E(\Phi_t)=\lim_{K\to\infty}\beta t(\sum_{j=0}^{K}\frac{1}{j!}\rho^j)^{-1}\frac{1}{K!}\rho^K=c>0,$$

then, as $K\to\infty$, $d_{TV}(\mathcal{L}(\Phi_t),\mathrm{POIS}(\lambda\nu_\theta))=\mathrm{O}(\frac{\log K}{K!}\rho^K)$, and $\mathrm{POIS}(\lambda\nu_\theta)$ converges weakly to $\mathrm{Po}(c)$. To prove this, note the following facts:

$$\sum_{i=0}^{K}\frac{i!}{K!}\rho^{K-i}-1=\frac{\rho}{K}(1+\frac{\rho}{K-1}+...+\frac{\rho^{K-1}}{(K-1)!})\leq\frac{\rho}{K}e^\rho;$$

$$\sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}\frac{i!}{j!}\rho^{j-i}=\sum_{j=1}^{K-1}\sum_{i=0}^{j-1}\frac{i!}{j!}\rho^{j-i}=\rho\sum_{j=1}^{K-1}\frac{1}{j}+\sum_{j=2}^{K-1}\sum_{i=0}^{j-2}\frac{i!}{j!}\rho^{j-i};$$

$$\sum_{j=2}^{K-1}\sum_{i=0}^{j-2}\frac{i!}{j!}\rho^{j-i}=\sum_{j=2}^{K-1}\frac{\rho^2}{j(j-1)}(1+\frac{\rho}{j-2}+...+\frac{\rho^{j-2}}{(j-2)!})<\infty;$$

$$\sum_{r=1}^{K}\frac{1}{r!}\rho^r\sum_{i=0}^{r-1}\sum_{j=i+1}^{K}\frac{i!}{j!}\rho^{j-i}=\sum_{r=1}^{K}\frac{1}{r!}\rho^r\sum_{j=1}^{K}\sum_{i=0}^{(j-1)\wedge(r-1)}\frac{i!}{j!}\rho^{j-i}$$

$$= \sum_{r=1}^{K} \frac{1}{r!}\rho^r \sum_{j=1}^{r}\sum_{i=0}^{j-1} \frac{i!}{j!}\rho^{j-i} + \sum_{j=2}^{K} \frac{1}{j!}\rho^j \sum_{r=1}^{j-1}\sum_{i=0}^{r-1} \frac{i!}{r!}\rho^{r-i} < \infty.$$

4. The number of lost customers among the first n

Theorem 4.1. *Let ω be the number of customers immediately before successive arrivals of new customers in a $GI/M/m/K$ queueing system ($m \le K$) with inter-arrival time distribution F_A and service intensity δ. Assume that the Markov chain ω is stationary with stationary distribution μ, and denote by ω^R the reverse Markov chain corresponding to ω. Let Ψ_n be the number of lost customers among the first n customers to arrive. Then,*

$$d_{TV}(\mathcal{L}(\Psi_n), \mathrm{POIS}(\lambda\nu_\theta)) \le H(\lambda,\theta) 3n\mu(K)^2 \Big(E(\tau_0(\omega^R)|\omega_0 = K)$$

$$+E(\tau_0(\omega)|\omega_0 = K) + 2E(\bar{\tau}_0(\omega))\Big) + 2P(\bar{\tau}_K(\omega) < \bar{\tau}_0(\omega)),$$

where $\nu_\theta(k) = (1-\theta)^{k-1}\theta \quad \forall k \in Z'_+$, $\lambda = n\mu(K)P(\tau_0(\omega) < \tau_K(\omega)|\omega_0 = K)$, $\theta = P(\tau_0(\omega) < \tau_K(\omega)|\omega_0 = K)$, and

$$H(\lambda,\theta) \le \begin{cases} (\frac{1}{\lambda\theta} \wedge 1)\exp(\lambda), & \text{if } \theta \in (0,1); \\ \frac{1}{\lambda\theta(2\theta-1)}\Big(\frac{1}{4\lambda\theta(2\theta-1)} + \log^+(2\lambda\theta(2\theta-1))\Big) \wedge 1, & \text{if } \theta \in [\frac{1}{2},1); \\ \frac{\theta^2}{\lambda(5\theta-4)}, & \text{if } \theta \in (\frac{4}{5},1). \end{cases}$$

Moreover, all quantities appearing in the bound can be calculated by solving four systems of linear equations of dimension $\le K$.

Proof. Consider a $GI/M/m/K$ queueing system to which the first customer arrives at time $t = 0$. Define the random sequence $\{\omega_t^+; t \in Z_+\}$ as the number of customers in the system immediately before each successive arrival of a new customer ($\omega_0^+ = 0$). It is well-known, see e.g. Section XI.3 in [1], that ω^+ is a Markov chain with transition matrix p defined by

$$p_{i,j} := \begin{cases} \int_{R_+} \binom{i+1}{i+1-j}(1-e^{-\delta t})^{i+1-j}e^{-\delta t j}dF_A(t), & \text{if } j \le i+1 \le m; \\ \int_{R_+}\int_0^t \binom{m}{m-j}(1-e^{-\delta(t-y)})^{m-j}e^{-\delta(t-y)j} & \\ \quad\times(\delta m)^{i+1-m}\frac{y^{i-m}}{(i-m)!}e^{-\delta m y}dydF_A(t), & \text{if } j < m < i+1 \le K; \\ \int_{R_+} e^{-\delta m t}\frac{(\delta m t)^{i+1-j}}{(i+1-j)!}dF_A(t), & \text{if } m \le j \le i+1 \le K; \\ p_{K-1,j}, & \text{if } i = K; \\ 0, & \text{if } j > i+1. \end{cases}$$

ω^+ is regenerative with regeneration times $\{t \in Z_+; \omega_t^+ = 0\}$. Since $E(\tau_0(\omega^+)) < \infty$, there exists a random sequence $\{\omega_t : t \in Z\}$ which is a stationary version of ω^+ (with index set Z). Clearly,

$$\Psi_n = \mathrm{card}\{t \in \{1,...,n\}; \omega_t = K\} \qquad \forall n \in Z_+.$$

In the same way as in the proof of Theorem 3.1, define the random sequence $\{(X_i,Y_i); i \in Z\}$ as follows. Let $\{X_i; i \in Z\}$ (where $... < X_{-1} < X_0 \le 0 < X_1 < ...$) be the random times $\{t \in Z; \omega_t = 0\}$, and let

$$Y_i := \mathrm{card}\{t \in Z; \omega_t = K, X_i \le t < X_{i+1}\} \qquad \forall i \in Z.$$

Define the random element ξ in $\mathcal{N}(Z, Z_+)$ as the point process generated by $\{(X_i, Y_i); i \in Z\}$, and define the random element $(\omega^o, \xi^o) \in \{0, ..., K\}^Z \times \mathcal{N}(Z \times Z_+)$ as a Palm version of (ω, ξ). Define the random sequence $\{(X_i^o, Y_i^o); i \in Z\}$ as the coordinates of the points of ξ^o, and define $\{T_i^o; i \in Z\}$ by $T_i^o := X_{i+1}^o - X_i^o \ \ \forall i \in Z$. The Palm inversion formula tells us that for each measurable function $g : \{0, ..., K\}^Z \times \mathcal{N}(Z \times Z_+) \to R_+$, it holds that

$$E(g(\omega, \xi)) = \frac{E(\sum_{i=0}^{T_0^o - 1} g(\theta_i(\omega^o, \xi^o)))}{E(T_0^o)}, \tag{4.1}$$

where $\theta : Z \times \{0, ..., K\}^Z \times \mathcal{N}(Z \times Z_+) \to \{0, ..., K\}^Z \times \mathcal{N}(Z \times Z_+)$ is the shift operator. With the notation $\mu_Y := \mathcal{L}(Y_0^o)$ and $\mu_Y' := \mu_Y(\cdot \cap Z_+')$, the triangle inequality implies that

$$d_{TV}(\mathcal{L}(\Psi_n), \mathrm{POIS}(\frac{n\mu_Y'}{E(T_0^o)})) \leq d_{TV}(\mathcal{L}(\Psi_n), \mathcal{L}(\int_{(0,n]\times Z_+'} v d\xi(u, v)))$$

$$+ d_{TV}(\mathcal{L}(\int_{(0,n]\times Z_+'} v d\xi(u, v)), \mathrm{POIS}(\frac{n\mu_Y'}{E(T_0^o)})),$$

and the basic coupling inequality implies that $d_{TV}(\mathcal{L}(\Psi_n), \mathcal{L}(\int_{(0,n]\times Z_+'} v d\xi(u, v))) \leq 2P(\bar{\tau}_K(\omega) < \bar{\tau}_0(\omega))$. For the second term, since ξ is a stationary (version of a) renewal reward process in the sense of Definition 4.1 in [4], Theorem 5.1 in [4] gives the bound

$$\begin{aligned} &d_{TV}(\mathcal{L}(\int_{(0,n]\times Z_+'} v d\xi(u, v)), \mathrm{POIS}(\frac{n\mu_Y'}{E(T_0^o)})) \\ &\leq H(\frac{n\mu_Y'}{E(T_0^o)}) \frac{3nE(Y_0^o)}{E(T_0^o)} \Big(\frac{E(T_0^o Y_0^o)}{E(T_0^o)} + \frac{E(T_0^o(T_0^o - 1))E(Y_0^o)}{E(T_0^o)^2} \Big), \end{aligned} \tag{4.2}$$

where $H(E(T_0^o)^{-1} n\mu_Y') \leq ((E(T_0^o)^{-1} n\mu_Y(1))^{-1} \wedge 1) \exp(E(T_0^o)^{-1} n\mu_Y(Z_+'))$, unless $\{k\mu_Y(k); k \in Z_+'\}$ is monotonically decreasing towards 0, in which case

$$H(\frac{n\mu_Y'}{E(T_0^o)}) \leq \frac{1}{\Delta_Y(1)} \Big(\frac{1}{4\Delta_Y(1)} + \log^+ 2\Delta_Y(1) \Big) \wedge 1,$$

where $\Delta_Y(1) := E(T_0^o)^{-1} n(\mu_Y(1) - 2\mu_Y(2))$. Again, Theorem 2.5 in [2] tells us that if

$$\kappa := \frac{\sum_{k=2}^\infty k(k-1)\mu_Y(k)}{\sum_{k=1}^\infty k\mu_Y(k)} < \frac{1}{2},$$

then it also holds that

$$H(\frac{n\mu_Y'}{E(T_0^o)}) \leq \frac{E(T_0^o)}{n(1 - 2\kappa) \sum_{k=1}^\infty k\mu_Y(k)}.$$

From (4.1) we get that $E(T_0^o)^{-1} = \mu(0)$ and that $E(T_0^o)^{-1} E(Y_0^o) = \mu(K)$, where μ is the stationary distribution of ω. We likewise get that

$$\frac{E(T_0^o(T_0^o - 1))}{2E(T_0^o)} = E(\bar{\tau}_0(\omega)),$$

and that

$$\frac{E(T_0^o Y_0^o)}{E(T_0^o)} = \mu(K)(E(\tau_0(\omega^R)|\omega_0 = K) + E(\tau_0(\omega)|\omega_0 = K)).$$

The strong Markov property implies that $\mu_Y(k) = P(\tau_K(\omega) < \tau_0(\omega)|\omega_0 = 0)(1-\theta)^{k-1}\theta \ \ \forall k \in Z'_+$, where $\theta = P(\tau_0(\omega) < \tau_K(\omega)|\omega_0 = K)$. From this it follows that $\mathrm{POIS}(E(T_0^\circ)^{-1}n\mu'_Y) = \mathrm{POIS}(\lambda\nu_\theta)$, where $\lambda = E(T_0^\circ)^{-1}nP(\tau_K(\omega) < \tau_0(\omega)|\omega_0 = 0)$, and that $H(\lambda,\theta) := H(E(T_0^\circ)^{-1}n\mu'_Y)$ satisfies the inequalities claimed in Theorem 4.1.

The last assertion of the theorem is a consequence of the following well-known facts. For each disjoint $A \subset \{0,...,K\}$ and $B \subset \{0,...,K\}$, the function $f : (A \cup B)^c \to [0,1]$ defined by $f(j) := P(\bar\tau_A(\omega) < \bar\tau_B(\omega)|\omega_0 = j) \ \ \forall j \in (A\cup B)^c$ is the unique solution of the system of linear equations $f(j) - \sum_{k=0}^K p_{j,k}f(k) = 0 \ \ \forall j \in (A\cup B)^c$ with boundary values $f(j) = I_A(j) \ \ \forall j \in A\cup B$. Similarly, the function $f_1 : A^c \to R_+$ defined by $f_1(j) := E(\bar\tau_A(\omega)|\omega_0 = j) \ \ \forall j \in A^c$ is the unique solution of $f_1(j) - \sum_{k=0}^K p_{j,k}f_1(k) = 1 \ \ \forall j \in A^c$ with boundary values $f_1(j) = 0 \ \ \forall j \in A$, and the function $f_2 : A^c \to R_+$ defined by $f_2(j) := E(\bar\tau_A(\omega^R)|\omega_0 = j) \ \ \forall j \in A^c$ is the unique solution of $f_2(j) - \sum_{k=0}^K \frac{\mu(k)}{\mu(j)}p_{k,j}f_2(k) = 1 \ \ \forall j \in A^c$ with boundary values $f_2(j) = 0 \ \ \forall j \in A$. $\square$

Theorem 4.2. *Let ω be the number of customers immediately before successive arrivals of new customers in a $M/M/m/K$ queueing system ($m \le K$) with arrival intensity β and service intensity δ. Assume that ω is stationary, and let Ψ_n be the number of lost customers among the first n customers to arrive. Let $\sigma_0 := 1$ and $\sigma_j := \prod_{i=1}^j (i\wedge m) \ \ \forall j \in \{1,...,K\}$. Then,*

$$d_{TV}(\mathcal{L}(\Psi_n), \mathrm{POIS}(\lambda\nu_\theta)) \le H(\lambda,\theta)3n(\sum_{j=0}^K \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-2}\frac{1}{\sigma_K^2}(\frac{\beta}{\delta})^{2K}$$

$$\times\Big(K+1+2\sum_{i=0}^{K-1}\sum_{j=i+1}^K \frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i} + 2\big(\sum_{j=0}^K \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j\big)^{-1}\sum_{r=1}^{K-1}\frac{1}{\sigma_r}(\frac{\beta}{\delta})^r\sum_{i=0}^r\sum_{j=i+1}^K\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}$$

$$+2\big(\sum_{j=0}^K \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j\big)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\sum_{i=0}^{K-1}\sum_{j=i+1}^K\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i} + 2\big(\sum_{j=0}^K \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j\big)^{-1}\sum_{j=1}^K\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j\Big)$$

$$+2(\sum_{j=0}^K \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\Big(1+(\sum_{i=0}^K\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1}\sum_{i=0}^{K-2}\sum_{j=i}^{K-1}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}\Big),$$

where $\nu_\theta(k) = (1-\theta)^{k-1}\theta \ \ \forall k \in Z'_+$, and:

$$\lambda = n(\sum_{j=0}^K\frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K(\sum_{i=0}^K\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1}; \quad \theta = (\sum_{i=0}^K\frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1};$$

$$H(\lambda,\theta) \le \begin{cases} (\frac{1}{\lambda\theta}\wedge 1)\exp(\lambda), & \text{if } \theta \in (0,1);\\ \frac{1}{\lambda\theta(2\theta-1)}\Big(\frac{1}{4\lambda\theta(2\theta-1)} + \log^+(2\lambda\theta(2\theta-1))\Big)\wedge 1, & \text{if } \theta \in [\frac{1}{2},1);\\ \frac{\theta^2}{\lambda(5\theta-4)}, & \text{if } \theta \in (\frac{4}{5},1). \end{cases}$$

Proof. This is a special case of Theorem 4.1 where the interarrival time distribution is $\exp(\beta^{-1})$. In this case it is straightforward to show, using partial integrations, that the transition matrix p of the Markov chain ω is given by

$$p_{i,j} := \begin{cases} \frac{\beta}{\beta+(j\wedge m)\delta}\prod_{r=j+1}^{i+1}\frac{(r\wedge m)\delta}{\beta+(r\wedge m)\delta}, & \text{if } j \leq i+1 \leq K; \\ p_{K-1,j}, & \text{if } i = K; \\ 0, & \text{if } j > i+1. \end{cases}$$

All quantities appearing in the total variation distance bound can be explicitly computed, as follows. Let the random element η^+ in $\mathcal{D}(R_+, \{0, ..., K+2\})$ be the Markovian pure jump process defined in the proof of Theorem 3.1. Let $\{\chi_t^+; t \in Z_+\}$ be the embedded Markov chain at jump times of η^+, and let $\{\chi_t; t \in Z\}$ be a stationary version of χ^+ (with index set Z). Define $\{\chi_t^{K+1}; t \in Z\}$ and $\{\chi_t^K; t \in Z\}$ by $\chi_i^{K+1} := \chi_i \wedge (K+1) \quad \forall i \in Z$ and $\chi_i^K := \chi_i \wedge K \quad \forall i \in Z$. Clearly, χ^K is a stationary birth-death chain with a transition matrix p^K given by

$$p_{i,j}^K := \begin{cases} 1, & \text{if } i = 0 \text{ and } j = 1; \\ \frac{\beta}{\beta+(i\wedge m)\delta}, & \text{if } 1 \leq i \leq K-1 \text{ and } j = i+1; \\ \frac{\beta}{\beta+m\delta}, & \text{if } i = j = K; \\ \frac{(i\wedge m)\delta}{\beta+(i\wedge m)\delta}, & \text{if } 1 \leq i \leq K \text{ and } j = i-1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{U_i; i \in Z\}$ (where $... < U_{-1} < U_0 \leq 0 < U_1 < ...$) be the random times $\{i \in Z; \chi_i^K \leq \chi_{i+1}^K\}$. It holds that $\mathcal{L}(\chi_{U_i}^K; i \in Z | U_0 = 0) = \mathcal{L}(\omega)$. The stationarity follows from the stationarity of χ^K, and the fact that the Markov property holds with transition matrix p can be shown using the fact that χ^K is a birth-death chain with transition matrix p^K. This embedding enables us to compute all the quantities we need. First, let μ^K be the stationary distribution of χ^K. It follows from (2.1) that the stationary distribution μ of ω is given by

$$\mu(r) = \frac{P(\chi_0^K = r, \chi_1^K = (r+1)\wedge K)}{P(\chi_0^K \leq \chi_1^K)} = \frac{\mu^K(r)p_{r,(r+1)\wedge K}^K}{\sum_{j=0}^K \mu^K(j)p_{j,(j+1)\wedge K}^K}$$

$$= \Big(\sum_{j=0}^K \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j\Big)^{-1}\frac{1}{\sigma_r}(\frac{\beta}{\delta})^r \qquad \forall r \in \{0, ..., K\}.$$

Next, using the embedding of χ^K into χ^{K+1} and the fact that χ^{K+1} is "almost" a birth-death chain, we get:

$$P(\tau_0(\omega) < \tau_K(\omega)|\omega_0 = r) = P(\tau_0(\chi^{K+1}) < \tau_{K+1}(\chi^{K+1})|\chi_0^{K+1} = r+1)$$

$$= \Big(\sum_{j=0}^K \sigma_j(\frac{\delta}{\beta})^j\Big)^{-1}\sum_{i=r+1}^K \sigma_i(\frac{\delta}{\beta})^i \qquad \forall r \in \{0, ..., K-1\},$$

and

$$P(\tau_0(\omega) < \tau_K(\omega)|\omega_0 = K) = P(\tau_0(\omega) < \tau_K(\omega)|\omega_0 = K-1)$$

$$= \Big(\sum_{j=0}^K \sigma_i(\frac{\delta}{\beta})^i\Big)^{-1}\sigma_K(\frac{\delta}{\beta})^K = \Big(\sum_{i=0}^K \frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i}\Big)^{-1}.$$

This also implies:

$$P(\bar{\tau}_K(\omega) < \bar{\tau}_0(\omega)) = \mu(K) + \sum_{r=1}^{K-1} \mu(r)(\sum_{i=0}^{K} \sigma_i(\frac{\delta}{\beta})^i)^{-1} \sum_{j=0}^{r} \sigma_j(\frac{\delta}{\beta})^j$$

$$= (\sum_{j=0}^{K} \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\Big(\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K + (\sum_{i=0}^{K} \sigma_i(\frac{\delta}{\beta})^i)^{-1} \sum_{i=0}^{K-1}\sum_{j=i\vee 1}^{K-1} \frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}\Big)$$

$$= (\sum_{j=0}^{K} \frac{1}{\sigma_j}(\frac{\beta}{\delta})^j)^{-1}\frac{1}{\sigma_K}(\frac{\beta}{\delta})^K\Big(1 + (\sum_{i=0}^{K} \frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i})^{-1} \sum_{i=0}^{K-2}\sum_{j=i}^{K-1} \frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}\Big).$$

It remains to calculate $E(\tau_0(\omega)|\omega_0 = K)$, $E(\tau_0(\omega^R)|\omega_0 = K)$ and $E(\bar{\tau}_0(\omega))$. From considerations of the sample paths of χ^K, and the embedding of χ^K into χ^{K+1}, we get:

$$E(\tau_0(\omega)|\omega_0 = r) = \frac{1}{2}\Big(E(\tau_0(\chi^K)|\chi_0^K = r+1) - (r+1)$$

$$-E(\sum_{i=1}^{\tau_0(\chi^{K+1})-1} I\{\chi_i^{K+1} = K+1\}|\chi_0^{K+1} = r+1)\Big)$$

$$+E(\sum_{i=1}^{\tau_0(\chi^{K+1})-1} I\{\chi_i^{K+1} = K+1\}|\chi_0^{K+1} = r+1) + 1 \qquad \forall r \in \{0, ..., K-1\}.$$

(2.3) implies that

$$E(\tau_0(\chi^K)|\chi_0^K = r+1) = \sum_{i=0}^{r}\sum_{j=i+1}^{K} \frac{\beta + (j\wedge m)\delta}{\delta}\frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i-1}$$

$$= \sum_{i=0}^{r}\sum_{j=i+1}^{K} \frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i} + \sum_{i=0}^{r}\sum_{j=i}^{K-1} \frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i}$$

$$= 2\sum_{i=0}^{r}\sum_{j=i+1}^{K} \frac{\sigma_i}{\sigma_j}(\frac{\beta}{\delta})^{j-i} + r + 1 - \sum_{i=0}^{r} \frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i},$$

while the strong Markov property for χ^{K+1} together with (2.2) implies that

$$E(\sum_{i=1}^{\tau_0(\chi^{K+1})-1} I\{\chi_i^{K+1} = K+1\}|\chi_0^{K+1} = r+1)$$

$$= \frac{P(\tau_{K+1}(\chi^{K+1}) < \tau_0(\chi^{K+1})|\chi_0^{K+1} = r+1)}{P(\tau_0(\chi^{K+1}) < \tau_{K+1}(\chi^{K+1})|\chi_0^{K+1} = K+1)}$$

$$= \frac{\beta + m\delta}{m\delta}\frac{P(\tau_{K+1}(\chi^{K+1}) < \tau_0(\chi^{K+1})|\chi_0^{K+1} = r+1)}{P(\tau_0(\chi^{K+1}) < \tau_{K+1}(\chi^{K+1})|\chi_0^{K+1} = K-1)}$$

$$= \frac{\beta + m\delta}{m\delta}\frac{1}{\sigma_{K-1}}(\frac{\beta}{\delta})^{K-1}(1 + \frac{m\delta}{\beta})^{-1}\sum_{j=0}^{r} \sigma_j(\frac{\delta}{\beta})^j = \sum_{i=0}^{r} \frac{\sigma_i}{\sigma_K}(\frac{\beta}{\delta})^{K-i}.$$

Summing up, we get:

$$E(\tau_0(\omega)|\omega_0 = r) = \sum_{i=0}^{r} \sum_{j=i+1}^{K} \frac{\sigma_i}{\sigma_j} (\frac{\beta}{\delta})^{j-i} + 1 \qquad \forall r \in \{0, ..., K-1\};$$

$$E(\tau_0(\omega)|\omega_0 = K) = E(\tau_0(\omega)|\omega_0 = K-1) = \sum_{i=0}^{K-1} \sum_{j=i+1}^{K} \frac{\sigma_i}{\sigma_j} (\frac{\beta}{\delta})^{j-i} + 1.$$

This also implies:

$$E(\bar{\tau}_0(\omega)) = \Big(\sum_{j=0}^{K} \frac{1}{\sigma_j} (\frac{\beta}{\delta})^j\Big)^{-1} \Big(\sum_{r=1}^{K-1} \frac{1}{\sigma_r} (\frac{\beta}{\delta})^r \sum_{i=0}^{r} \sum_{j=i+1}^{K} \frac{\sigma_i}{\sigma_j} (\frac{\beta}{\delta})^{j-i}$$

$$+ \frac{1}{\sigma_K} (\frac{\beta}{\delta})^K \sum_{i=0}^{K-1} \sum_{j=i+1}^{K} \frac{\sigma_i}{\sigma_j} (\frac{\beta}{\delta})^{j-i} + \sum_{j=1}^{K} \frac{1}{\sigma_j} (\frac{\beta}{\delta})^j\Big).$$

Finally, from considerations of the sample paths of χ^K, the reversibility of χ^K, and the embedding of χ^K into χ^{K+1}, we get:

$$E(\tau_0(\omega^R)|\omega_0 = r) = \frac{1}{2}\Big(E(\tau_0(\chi^K)|\chi_0^K = r) - r$$

$$-E(\sum_{i=1}^{\tau_0(\chi^{K+1})-1} I\{\chi_i^{K+1} = K+1\}|\chi_0^{K+1} = r)\Big)$$

$$+E(\sum_{i=1}^{\tau_0(\chi^{K+1})-1} I\{\chi_i^{K+1} = K+1\}|\chi_0^{K+1} = r) + r \qquad \forall r \in \{1, ..., K\},$$

and combining this with the previous results gives

$$E(\tau_0(\omega^R)|\omega_0 = r) = \sum_{i=0}^{r-1} \sum_{j=i+1}^{K} \frac{\sigma_i}{\sigma_j} (\frac{\beta}{\delta})^{j-i} + r \qquad \forall r \in \{1, ..., K\}.$$

□

Example 4.1. Let Ψ_n be the number of lost customers among n successively arriving customers in a stationary $M/M/1/K$ queueing system with arrival intensity β and service intensity δ, and let $\rho = \beta/\delta < 1$. Then,

$$d_{TV}(\mathcal{L}(\Psi_n), \mathrm{POIS}(\lambda\nu_\theta)) \leq H(\lambda, \theta) 3n \frac{(1-\rho)^2}{(1-\rho^{K+1})^2} \rho^{2K}$$

$$\times\Big(K + 1 + \frac{2\rho}{1-\rho}(K - \frac{\rho(1-\rho^K)}{1-\rho}) + \frac{2\rho^{K+1}}{1-\rho^{K+1}}(K - \frac{\rho(1-\rho^K)}{1-\rho})$$

$$+\frac{2\rho^2(2 - \rho - \rho^K + \rho^{K+1} - \rho^{2K} - 2K(1-\rho)\rho^{K-1})}{(1-\rho)^2(1-\rho^{K+1})} + \frac{2\rho(1-\rho^K)}{1-\rho^{K+1}}\Big)$$

$$+\frac{2(1-\rho)\rho^K}{1-\rho^{K+1}}\Big(1 + \frac{1}{1-\rho^{K+1}}(K - 1 - \frac{\rho^2(1-\rho^{K-1})}{1-\rho})\Big),$$

where

$$\lambda = n\frac{(1-\rho)^2}{(1-\rho^{K+1})^2}\rho^K; \qquad \theta = \frac{1-\rho}{1-\rho^{K+1}}.$$

In particular, if $\rho < 1$ is constant and $n := n(K)$ is chosen so that

$$\lim_{K\to\infty} E(\Psi_n) = \lim_{K\to\infty} n\frac{1-\rho}{1-\rho^{K+1}}\rho^K = c > 0,$$

then, as $K \to \infty$, $d_{TV}(\mathcal{L}(\Psi_n), \mathrm{POIS}(\lambda\nu_\theta)) = \mathrm{O}(K\rho^K)$, and $\mathrm{POIS}(\lambda\nu_\theta)$ converges weakly to $\mathrm{POIS}(c(1-\rho)\nu_{(1-\rho)})$.

Example 4.2. Let Ψ_n be the number of lost customers among n successively arriving customers in a stationary $M/M/K/K$ queueing system with arrival intensity β and service intensity δ. If $\rho = \beta/\delta$ is constant and $n := n(K)$ is chosen so that

$$\lim_{K\to\infty} E(\Psi_n) = \lim_{K\to\infty} n\Big(\sum_{j=0}^{K}\frac{1}{j!}\rho^j\Big)^{-1}\frac{1}{K!}\rho^K = c > 0,$$

then, as $K \to \infty$, $d_{TV}(\mathcal{L}(\Psi_n), \mathrm{POIS}(\lambda\nu_\theta)) = \mathrm{O}(\frac{K}{K!}\rho^K)$, and $\mathrm{POIS}(\lambda\nu_\theta)$ converges weakly to $\mathrm{Po}(c)$. This follows in the same way as in Example 3.2, using also the following facts:

$$\sum_{i=0}^{K-2}\sum_{j=i}^{K-1}\frac{i!}{j!}\rho^{j-i} = K-1+\sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}\frac{i!}{j!}\rho^{j-i};$$

$$\sum_{r=1}^{K-1}\frac{1}{r!}\rho^r\sum_{i=0}^{r}\sum_{j=i+1}^{K}\frac{i!}{j!}\rho^{j-i} = \sum_{r=1}^{K-1}\frac{1}{r!}\rho^r\sum_{j=1}^{K}\sum_{i=0}^{(j-1)\wedge r}\frac{i!}{j!}\rho^{j-i}$$

$$= \sum_{r=1}^{K-1}\frac{1}{r!}\rho^r\sum_{j=1}^{r}\sum_{i=0}^{j-1}\frac{i!}{j!}\rho^{j-i} + \sum_{j=2}^{K}\frac{1}{j!}\rho^j\sum_{r=1}^{j-1}\sum_{i=0}^{r}\frac{i!}{r!}\rho^{r-i} < \infty.$$

5. Numerical calculations

In the final section we have numerically evaluated the total variation distance bounds given in Theorems 3.1 and 4.2, for some typical values of the parameters β, δ, m, K and t (or n). The corresponding values of λ/t and θ have been computed as well. For the bound given in Theorem 3.1, the values can be found in Table 5.1 (where $\beta = 1$, $\delta = 4$ and $m = 1$) and Table 5.2 (where $\beta = 1$, $\delta = 0.5$ and $m = K$). For the bound given in Theorem 4.2 the values can be found in Table 5.3 ($\beta = 1$, $\delta = 4$ and $m = 1$) and Table 5.4 ($\beta = 1$, $\delta = 0.5$ and $m = K$).

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K	λ/t	θ	t			
			10^2	10^4	10^6	10^8
4	0.002202	0.7507	0.02391	0.5095	>1	>1
6	$1.373 \cdot 10^{-4}$	0.7500	$9.765 \cdot 10^{-4}$	0.007947	0.06432	0.1272
8	$8.583 \cdot 10^{-6}$	0.7500	$7.730 \cdot 10^{-5}$	$1.097 \cdot 10^{-4}$	0.002050	0.006657
10	$5.364 \cdot 10^{-7}$	0.7500	$6.084 \cdot 10^{-6}$	$6.231 \cdot 10^{-6}$	$2.092 \cdot 10^{-5}$	$2.795 \cdot 10^{-4}$

TABLE 5.1. The bound given in Theorem 3.1 ($\beta = 1$, $\delta = 4$, $m = 1$).

K	λ/t	θ	t			
			10^2	10^4	10^6	10^8
6	0.007062	0.5844	0.02218	0.6808	>1	>1
8	$6.263 \cdot 10^{-4}$	0.7287	$3.797 \cdot 10^{-4}$	0.007214	0.02728	0.04787
10	$3.029 \cdot 10^{-5}$	0.7930	$1.250 \cdot 10^{-5}$	$3.124 \cdot 10^{-5}$	$4.637 \cdot 10^{-4}$	0.001081
12	$9.605 \cdot 10^{-7}$	0.8300	$3.694 \cdot 10^{-7}$	$3.870 \cdot 10^{-7}$	$2.143 \cdot 10^{-6}$	$8.861 \cdot 10^{-6}$

TABLE 5.2. The bound given in Theorem 3.1 ($\beta = 1$, $\delta = 0.5$, $m = K$).

K	λ/n	θ	n			
			10^2	10^4	10^6	10^8
4	0.002202	0.7507	0.05772	0.7712	>1	>1
6	$1.373 \cdot 10^{-4}$	0.7500	0.003685	0.01530	0.1093	0.2140
8	$8.583 \cdot 10^{-6}$	0.7500	$3.049 \cdot 10^{-4}$	$3.633 \cdot 10^{-4}$	0.003856	0.01215
10	$5.364 \cdot 10^{-7}$	0.7500	$2.411 \cdot 10^{-5}$	$2.439 \cdot 10^{-5}$	$5.224 \cdot 10^{-5}$	$5.426 \cdot 10^{-4}$

TABLE 5.3. The bound given in Theorem 4.2 ($\beta = 1$, $\delta = 4$, $m = 1$).

K	λ/n	θ	n			
			10^2	10^4	10^6	10^8
8	$6.263 \cdot 10^{-4}$	0.7287	0.05770	0.9033	>1	>1
10	$3.029 \cdot 10^{-5}$	0.7930	0.002172	0.004567	0.05980	0.1386
12	$9.605 \cdot 10^{-7}$	0.8300	$6.927 \cdot 10^{-5}$	$7.158 \cdot 10^{-5}$	$3.029 \cdot 10^{-4}$	0.001188
14	$2.175 \cdot 10^{-8}$	0.8552	$1.621 \cdot 10^{-6}$	$1.622 \cdot 10^{-6}$	$1.739 \cdot 10^{-6}$	$1.200 \cdot 10^{-5}$

TABLE 5.4. The bound given in Theorem 4.2 ($\beta = 1$, $\delta = 0.5$, $m = K$).

Current address: Department of Mathematics, KTH, S-100 44 Stockholm, Sweden
E-mail address: ter@math.kth.se