Essay in Game Theory

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1 Introduction

This is an essay which aims to capture the most relevant aspects of the paper "Potential Games" by Dov Monderer and Lloyd S. Shapley. In this article Monderer and Shapley introduces potential games, games which admit a potential function. The potential function is a real valued function on the strategy space which match a deviation to a change of the potential value. The matching can be exact (*exact potential games*) or by sign (*ordinal potential games*). Nash equilibria are local maximizers of the potential function, and since deviations giving higher payoff increase the potential, best-reply dynamics converge to the Nash equilibrium.

2 Potential Games

In order to proceed to more interesting results a few definitions and basic results are necessary. The concepts of *ordinal* and *weighted* potential games are first introduced. Consider a game $\Gamma(u^1, u^2, \ldots, u^n)$ in strategic form with finite number of players, $N = \{1, 2, \ldots, n\}$. Let the strategy space of player i be given by Y^i and let $Y = Y^1 \times Y^2 \times \cdots \times Y^n$. The payoff function of player i is given by $u^i : Y \to R$. A function $P : Y \to R$ is an *ordinal* potential function if for every user i and any $y^{-i} \in Y$

$$u^i(y^{-i}, x_i) - u^i(y^{-i}, z_i) > 0, \quad \text{iff } P(y^{-i}, x_i) - P(y^{-i}, z_i) > 0$$

for every $x_i, z_i \in Y^i$. A game Γ is called an ordinal potential game if it has an ordinal potential function. Similarly *ω-potential* games are defined using weights $\omega^i, i \in N$ by the relation

$$u^i(y^{-i}, x_i) - u^i(y^{-i}, z_i) = \omega^i[P(y^{-i}, x_i) - P(y^{-i}, z_i)]$$

for every $x_i, z_i \in Y^i$. If there is a function P such that the relation is fulfilled with $\omega^i = 1, \forall i$ the game is called an (exact) potential game and the function P is called an (exact) potential function. Note that a difference between ordinal and exact potential games is that for ordinal potential games only the sign of the differences matters, whereas the differences are equal for exact potential games.

A basic result of ordinal potential games is that the set of equilibrium points of $\Gamma(u^1, u^2, \ldots, u^n)$ is equal to the equilibrium set of $\Gamma(P, P, \ldots, P)$. Hence $y \in Y$ is an equilibrium point of Γ iff for every $i \in N$

$$P(y) \geq P(y^{-i}, x), \quad \forall x \in Y^i.$$

Furthermore every finite ordinal game possesses a pure-strategy equilibrium.

Consider the following game

$$G = \begin{pmatrix} (1,1) & (9,0) \\ (0,9) & (6,6) \end{pmatrix}.$$

From the definition of potential we can verify that the following is a potential function of G

$$P = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix},$$

and hence G is a potential game.

A useful concept for the characterization of potential games is the notion of a *path.* A path is defined as a sequence of actions, $\gamma = \{\gamma_0, \gamma_1, \dots\}$, such that for every $\gamma_k, k \geq 1$ a unique player makes a change in its action. An improvement path is a path where the unique player changing its action obtains a strictly higher payoff, $u^i(y^k) > u^i(y^{k-1})$. A game has the finite improvement property (FIP) if every improvement path is finite. Every finite ordinal game has this property. For games with the FIP it holds that also the maximal improvement path (best response) ends up in the equilibrium and there is convergence for these dynamics.

For exact potential games the following result is presented. Let P_1 and P_2 be two potential functions for the game Γ. Then there exists a constant c s.t. $P_1(y) - P_2(y) = c$ for every $y \in Y$. This implies that the potential function of a game is unique up to an additive constant, which implies that for determining the equilibrium set it does not matter which potential function is used.

In the following potential games are related to the potential concept in physics. For a finite path γ and a vector of functions $v = (v_1, v_2, \dots, v_n)$ define

$$I(\gamma, v) = \sum_{k=1}^{n} [v^{i_k}(y_k) - v^{i_k}(y_{k-1})],$$

where i_k is the unique player deviating at step k. Hence we sum over the differences of the function values of the current and previous action for the unique deviating player, over a path. A path is called closed if it begins and ends in the same point ($\gamma_0 = \gamma_n$), and it is called a simple closed path if in addition the intermediate steps are not equal to eachother. The following useful theorem shows the similarities to the physical concept.

Let Γ be a game in strategic form. Then the following statements are equivalent.

- Γ is a potential game.
- $I(\gamma, u) = 0$ for every finite closed path γ.
- $I(\gamma, u) = 0$ for every finite simple closed path γ.
- $I(\gamma, u) = 0$ for every finite simple closed path γ of length 4.

This theorem can be used to determine whether a game is a potential game or not. It relates to psysics in that for a potential field any way between two points give the same change in potential.

3 Extensions

Now consider infinite games. A game Γ is a continuous game if the strategy sets are topological spaces and the payoff functions are continuous w.r.t. the product spaces. The continuity of the payoff function implies continuity of the potential function. For a continuous potential game with compact strategy sets it holds that it has a pure-strategy equilibrium point. If the payoff functions are continuously differentiable functionals, P is a potential function of the game iff P is continuously differentiable and the following relation holds

$$\frac{\partial u^i}{\partial y^i} = \frac{\partial P}{\partial y^i}, \quad \forall i \in N.$$

Furthermore, if the payoff functions are twice continuously differentaible it holds that the game is a potential game iff

$$\frac{\partial^2 u^i}{\partial y^i \partial y^j} = \frac{\partial^2 u^j}{\partial y^i \partial y^j}, \quad \forall i, j \in N.$$

A potential is then given by

$$P(y) = \sum_{i=1}^{n} \int_0^1 \frac{\partial u^i}{\partial y^i}(x(t))(x^i)'(t)dt$$

where $x : [0 \quad 1] \to Y$ is a piecewise continuously differentiable path in Y.

4 Congestion games

An important class of games is the class of congestion games. It has important applications in networks since it has players whose payoffs, or costs, depend on the congenstion, that is how many other players it shares resourses with. In the article it is shown that every congeston game is a potential game and every potential game is isomorphic to a congestion game.

5 Example

Consider a Cournot competition with cost functions $c_i(q) = cq_i, \quad \forall i$ and a positive inverse demand function $F(Q)$, where $Q = \sum_{i=1}^{n} q_i > 0$. Let the profit of player i be defined as

$$\Pi_i(q_1, \dots, q_n) = F(Q)q_i - cq_i.$$

Now define the function P by

$$P(q_1, \dots, q_n) = q_1 q_2 \dots q_n (F(Q) - c).$$

Then it holds that for every player i and q_{-i}

$$\Pi_i(q_i, q_{-i}) - \Pi_i(x_i, q_{-i}) > 0 \quad \text{iff} \quad P(q_i, q_{-i}) - P(x_i, q_{-i}) > 0,$$

for all $q_i, x_i \in R_{++}$. Hence P is an ordinal potential and the game is an ordinal potential game.

Referenser

[1] Dov Monderer and Lloyd S. Shapley (1994) Potential Games, *Games and economic behaviour* **14**, 124–143.