Large Robust Games
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The summary is of the article Large Robust Games by E. Kalai (2004).

1 The setting

Assume that we are about to analyse a game with a large number of players that shall choose an action each. There might be a couple of Nash equilibrium for this game, but if we start to look for Nash equilibrias to some extensive form of this game, the equilibrias are not necessarily preserved. In many cases, it can be hard to derive the exact form of the extensive game that is played, for example, it might be uncertainties regarding which player that gets the most information and in what order the players make their actions. This is why it would be appropriate to study at what occations a Nash equilibrium of some extensive form game is robust in the sense that it also is a Nash equilibrium in each extensive form of the game, which would imply that no extra care would be needed to analyse what extensive form the game acctually is.

1.1 The game settings

The article defines a game in the following way. $\mathcal{T}$ and $\mathcal{A}$ are two non empty sets containing player types and player actions that are possible during a game, let $\mathcal{K} := \mathcal{T} \times \mathcal{A}$ contain what is defined as type action characters. A family $\Gamma = \Gamma(\mathcal{T}, \mathcal{A})$ contains Bayesian games $G = (N, T, \tau, A, u)$ where $N = \{1, \dots, n\}$ are the players, $T = \times_{i=1}^{n} T_i$ are type profiles,where $T_i \subseteq \mathcal{T}$, $\forall i \in N$. $\tau = (\tau_1, \dots, \tau_n)$ are such that $\tau_i(t) \geq 0$, $\sum_{t \in T_i} \tau_i(t) = 1 \; \forall i \in N$, $A = \times_{i=1}^{n} A_i$ where $A_i \subseteq \mathcal{A}$, $\forall i \in N$ and $C_i := T_i \times A_i$ defines the possible type action characters for player i in game G. The utility function $u = (u_1, \dots, u_n)$ is then such that $u : C \to \times_{i=1}^{n} [0, 1]$. Furthermore, a Bayesian game, mentioned above,is defined such that each player i is randomly given a type $t_i \in T_i$ through a predetermined probability distribution τ_i, after which all players simultaneously, by choosing a pure or mixed strategy, choose an action $a_i \in A_i$ and recieves the payoff generated by $c = (\{t_1, a_1\}, \dots, \{t_n, a_n\}) \in C$.

From this process, we may derive $\gamma(c) = (P(c_1 = \{t_1, a_1\}), \dots, P(c_n = \{t_n, a_n\}))$, where $P(c_i = \{t_i, a_i\}) := \sigma_i(a_i|t_i)\tau_i(t_i)$, for a (pure or mixed) strategy σ_i. When the condition that the individuall probability distributions are independent we get $P(c) = \Pi_{i=n}^{n} \gamma_i(c_i)$. The utility function becomes $u_i(\sigma) = \mathbb{E}[u_i(c)]$ and a Nash equilibrium σ is a strategy such that $u_i(\sigma) \geq u_i(\sigma_i', \sigma_{-i})$ for all players and all strategies σ_i'.

1.2 Semi anonymity och continuity

Definition 1. For each $c \in C$ the empirical distribution is defined as

$$emp_c(\kappa) = \frac{\sum_{i=1}^{n} \mathbb{I}_{\{c_i = \kappa\}}}{n}.$$

Some limitations on the family Γ are needed for the results in the next sction, that is the reason for the next two definitions. The first definition says that the payoff for a player must not be dependent on the identity of the players,but only on the type action characters played, the other one gives a type of continuity with respect to the empirical distribution defined in Definition 1.

Definition 2. The games in Γ are *semi anonymous* if, for each player i and for each $c, \bar{c} \in C$ we have $u_i(c) = u_i(\bar{c})$ when $c_i = \bar{c}_i$ and $emp_{c_{-i}}(\kappa) = emp_{\bar{c}_{-i}}(\kappa)$.

Definition 3. The payoff function u is *uniformly equicountinous* if, for each $\epsilon > 0$ exists a $\delta > 0$ such that for all $G \in \Gamma$ and each player $i \in N$, it follows that $c, \bar{c} \in C$, $|u_i(c) - u_i(\bar{c}_i)| < \epsilon$ då $c_i = \bar{c}_i$ och $\max_{\kappa \in \mathcal{K}} |emp_{c_{-i}}(\kappa) - emp_{\bar{c}_{-i}}(\kappa)| < \delta$.

1.3 Ex-post Nash and extensively robust strategies

Two central concepts in the article are ex−post−Nash−strategies and extensively robust strategies. A Nash equilibrium to G is *ex−post−Nash* if every player, after all players simultaneously played their strategy and realised their action, is given the opportunity to look back at his strategy without wanting to change it, given the information of the other players actions. A Nash equilibrium to G is *extensively robust* if it i is a Nash equilibrium in each *extensive version* (defined below) of the game G. Take a Bayesiam game G.

One extensive version of G game would be to first let all players make their plays, and then let one of the players choose wheather to revise his previous play or not. A Nash equilibrium to this extensive version of G is ex−post−Nash per definition, but it is not necessarily extensively robust. The conclusion is that extensive robustness is a stronger condition than ex−post−Nash since an extensively robust strategy demands that the strategy is a Nash equilibrium in all extensive versions of G.

To handle the transition from ex−post−Nash to the much more involved extensive robustness, an *extensiv version* of the game G is defined, called $\bar{G}$. The idea is to make the possible designs of an extensive version so general that it may describe most of the twists that may be present in common extensive forms of G. A number of criterias are given for $\bar{G}$ to be an extensive version of G, with the important constraint that every *constant strategy* in G may be played. Define a strategy in $\bar{G}$ as $\bar{\sigma}$. A constant strategy $\bar{\sigma}$ is such that each player $i \in N$ has the possibility to choose all a_i that the player would be able to choose from in the game G in the first information set; the player then keep the action a_i in all later informationsets. This makes it possible to sustain the original action a_i independently of new information later on in the game.

2 Results

The first result needs the following definition.

Definition 4. Let $\epsilon > 0$. $c = (\{t_1, a_1\}, \ldots, \{t_n, a_n\})$ is $\epsilon-$*best response* for player i if all actions $a'_i \in A_i$ give $u_i(\{t_i, a'_i\}; c_{-i}) \leq u_i(c) + \epsilon$; c is $\epsilon-$Nash if this holds for each player $i \in N$.

Furthermore, for $\rho > 0$, a strategy profile σ is an (ϵ, ρ)*ex−post−Nash* equilibrium if the probability that the profile reach an $\epsilon-$Nash equilibrium of type action characters is at least $1 - \rho$.

The first result states that Nash equilibrias in a family of Bayesian games approaches an ex−post−Nash equilibrium in the approximate sense of Definition 4 as the number of players increases.

Teorem 1. Assume that the family of bayesian games $\Gamma(\mathcal{T}, \mathcal{A})$ are semi anonymous as in Definition 2 and that they have a continuos payoff function as in Definition 3 and let $\epsilon > 0$. then there exists constants $\alpha = \alpha(\Gamma, \epsilon)$ and $\beta = \beta(\Gamma, \epsilon)$ with $\beta < 1$ such that for each m we have that all Nash equilibrias to the games in Γ with at least m players are $(\epsilon, \alpha\beta^m)$ex−post−Nash.

For the other result we need some clarifications. For an extensive version $\bar{G}$, $\bar{\eta} = (\bar{\eta}_1, \ldots, \bar{\eta}_n)$ is a set of *behavioral strategies*. A player i is said to have a *better than $\epsilon-$improvement* in some information set A_i if there exists an $\bar{\eta}'$ such that $\mathbb{E}_{\bar{\eta}'}[u_i|A] - \mathbb{E}_{\bar{\eta}}[u_i|A] > \epsilon$, where $\bar{\eta}' = (\bar{\eta}_1, \ldots, \bar{\eta}'_i, \ldots, \bar{\eta}_n)$ and $\bar{\eta}'_i$ is such that $\bar{\eta}'_i$ and $\bar{\eta}_i$ coincides in all information sets belonging to player i that is not a follower of information set A.

Definition 5. A strategy profile $\bar{\eta}$ to $\bar{G}$ is an $(\epsilon, \rho)-$*Nash equilibrium* if the probability that a player has a better than $\epsilon-$improvement in any information set is not larger han ρ.

Definition 6. A Nash equilibrium σ to the Bayesian game G is $(\epsilon, \rho)-$*exstensively robust* if, in each extensive version $\bar{G}$ of G, it holds that the constant play version of σ, $\bar{\sigma}$, is an $(\epsilon, \rho)-$Nash equilibrium.

The following result is an extension of Theorem 1 such that it includes extensive versions.

Teorem 2. Assume that the family of bayesian games $\Gamma(\mathcal{T}, \mathcal{A})$ are semianonymous as in Definition 2 and that they have a continuous payoff function as in Definition 3 and let $\epsilon > 0$. Then there exists constants $\alpha = \alpha(\Gamma, \epsilon)$ and $\beta = \beta(\Gamma, \epsilon)$ with $\beta < 1$ such that all Nashequilibrias to game in Γ with at least m players are $(\epsilon, \alpha\beta^m)-$extensively robust.

3 Conclusions

As was noted in the beginning of this text, the result of the article in question is that, under some conditions, we might to some extent be able to neglect the issue of finding the correct form of the extensive form of a Bayesian game G when looking for a Nash equilibrium. The article shows that a large number of players gives a kind of stability in the Nash strategies that makes it less important to take other information into account than the form of G. This may be advantageous in situations where it is hard to determine what extensive form of the game G that is the correct one. An interesting example of this is given in the article, ; section 6.6 treats two theories concerning how to analyse trading.

One might argue that the agents on the market takes their actions based on market information, the agents own opinions and the price of the stock. The market is then in equilibrium since the value of the stock does not affect the actions of the agents.

One might also, from a game theory point of view, argue that some of this information is given to the agent *before* the choice to buy or sell is made, namely the opinions of the agent s and the market information, while some of the informationoch is given *after* the action is taken, namely the realised stockprice when the agent acctually bought or sold the stock, since the agent did not really know the price at which the trade was to be made in advance.

If the game for the agents fullfill the criterias in the theorems above, extensive robustness would imply that information about the realised price does not affect the decision of the agents, and therefore the two reasonings above yields the same Nash equilibrias.

References

[1] E. Kalai (2004). *"Large Robust Games"*. Econometrica, Vol. 72, No.6, p. 1631-1665