

Exercise 12.1 (The value of a European call option) *In the Black & Scholes model, the price S_t of a risky asset at time t is given by the formula:*

$$S_t = \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

The value of a European call option, with maturity T and strike price K , is $(S_T - K)^+$ at time T . If $T > t$, compute explicitly

$$E[(S_T - K)^+ | \mathcal{F}_t].$$

Solution: We have

$$\begin{aligned} E[(S_T - K)^+ | \mathcal{F}_t] &= E[(S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma B_T} - K)^+ | \mathcal{F}_t] \\ &= E[(S_0 e^{(r - \frac{1}{2}\sigma^2)(T-t+t) + \sigma(B_T - B_t + B_t)} - K)^+ | \mathcal{F}_t] \\ &= E[(S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)} - K)^+ | \mathcal{F}_t] \end{aligned}$$

To compute the expected value we need the Markov property of the Brownian motion (see *Proposition 12.4*). We start in a more general setting and apply the results to the expected value above.

Assume that B_t starts in x . Denoting the BM starting at x by $B^x(t)$ it is obvious that for each $t > 0$,

$$B^x(t) \in N(x, t).$$

The probability under which the Brownian motion starts at x , denoted by $\mathbf{P}^x$, is given by

$$\mathbf{P}^x(B_t \in A) = \mathbf{P}(B(t) \in A | B(0) = x) = \int_A p(t; x, y) dy$$

for every $A \in \mathcal{B}(\mathbf{R})$, where

$$p(t; x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right)$$

is the probability density of $B^x(t) \in N(x, t)$ of the Brownian motion. The expectation with respect to $\mathbf{P}^x$ is

$$E^x[h(B_t)] = \int_{\mathbf{R}} h(B_t) d\mathbf{P}^x = \int_{\mathbf{R}} h(y) p(t; x, y) dy.$$

Next, we consider a standard BM (starting at 0) and a nice function h , and we are interested in computing the expected value of $h(B_{t+s})$ given $\mathcal{F}_s$, i.e. we want to compute $E[h(B_{t+s}) | \mathcal{F}_s]$. Using the time-homogeneity of the Brownian motion we get

$$\begin{aligned} E[h(B_{t+s}) | \mathcal{F}_s] &= E[h(B_{t+s} - B_s + B_s) | \mathcal{F}_s] \\ &= \int_{\mathbf{R}} h(y) p(t; B_s, y) dy \\ &= E^{B_s}[h(B_t)] \end{aligned}$$

This is known as the *Markov property* of the Brownian motion and it states that any expectation of a function h of the Brownian motion evaluated at time T , $h(B_T)$, conditioned on a time $t < T$ is only dependent on the value B_t and the time $T - t$. Using the Markov property on the expectation above we get

$$\begin{aligned} \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] &= \mathbb{E}[(S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)} - K)^+ | \mathcal{F}_t] \\ &= E^{B_t}[(S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)} - K)^+] \end{aligned}$$

Next, we use the time-homogeneity of the Brownian motion to rewrite $B_T - B_t = \sqrt{T - t}X$, where $X \in N(0,1)$. Then

$$\begin{aligned} \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] &= E^{B_t}[(S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}X} - K)^+] \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}x} - K)^+ e^{-\frac{x^2}{2}} dx \end{aligned}$$

Note that the expression $S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}x} - K \geq 0$ when

$$x \geq \frac{\log(K/S_t) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} := d_1.$$

Hence, the integral can be rewritten as

$$\begin{aligned} \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] &= \frac{1}{\sqrt{2\pi}} \int_{d_1}^{\infty} (S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}x} - K) e^{-\frac{x^2}{2}} dx \\ &= S_t e^{r(T-t)} \int_{d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 - 2\sigma\sqrt{T-t}x + \sigma^2(T-t)}{2}} dx - K \underbrace{\int_{d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{=\mathbf{P}(X \geq d_1)} \\ &= S_t e^{r(T-t)} \int_{d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sigma\sqrt{T-t})^2}{2}} dx - K \mathbf{P}(X \geq d_1) \\ &= \{\text{Change of variables: } y = x - \sigma\sqrt{T-t}, dy = dx\} \\ &= S_t e^{r(T-t)} \underbrace{\int_{d_1 - \sigma\sqrt{T-t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{\mathbf{P}(X \geq d_1 - \sigma\sqrt{T-t})} - K \mathbf{P}(X \geq d_1) \\ &= S_t e^{r(T-t)} \mathbf{P}(X \geq d_1 - \sigma\sqrt{T-t}) - K \mathbf{P}(X \geq d_1). \end{aligned}$$

This is the explicit form of the call option price.