Hedging

Are options risky?

Consider a European call option on a underlying stock.

An extreme example:

Today's stock price = 100 SEK

Assume that the call option has strike price K=99 SEK and and should be exercised tomorrow.

$\Rightarrow$ the option price $\approx$ 1 SEK.

Assume that the stock price goes down 1 SEK until tomorrow.

The option then becomes worthless, i.e. the option price goes down 1 SEK.

This means that a drop of 1% in the stock price results in a drop of 100% in the option price!

Example

Assume that we have sold a European call option on 100 000 ABC stocks for 2 million SEK.

Data:
S=365, K=370, $\sigma = 0.2$, r=7%, $T-t$=1/4

The option price is given by

$$\begin{aligned} C(t,S) &= 100\,000\times \\ &\times\{SN[d_1(t,S)] - Ke^{-r(T-t)}N[d_2(t,S)]\} \\ &= 100\,000 \times 15.264 \end{aligned}$$

We seem to have made a profit of

$$2\,000\,000 - 1\,526\,400 = 473\,600 \text{ SEK}$$

but...

We are exposed to financial risk!

Strategies

Naked position: Do nothing!

Scenarios at time $t = T$

- If $S_T < 370$ the option will not be exercised.

  Our net earnings: 2 000 000 SEK.

- If $S_T > 370$ the option will be exercised.

  We will have to buy 100 000 stocks at the price S_T and sell them at the price 370 SEK

  Our net earnings:
  $2\,000\,000 - 100\,000 \times (S_T - 370)$ SEK

  - If S_T=385: 500 000 SEK
  - If S_T=395: $-500\,000$ SEK

Covered position: Buy 100 000 stocks now

Scenarios at time $t = T$

- If $S_T < 370$ the option will not be exercised.

  Our net earnings:
  $2\,000\,000 + 100\,000 \times (S_T - 365)$ SEK.

  - If S_T=360: 1 500 000 SEK
  - If S_T=340: $-$500 000 SEK

- If $S_T > 370$ the option will be exercised.

  Our net earnings:
  $2\,000\,000 + 100\,000 \times (370 - 365) =$
  $= 2\,500\,000$ SEK

Strategy 1 will always result in a profit if $S_T < K$, but can result in a loss if $S_T > K$.

Strategy 2 will always result in a profit if $S_T > K$, but can result in a loss if $S_T < K$.

Are there other possible strategies?

The purpose of hedging:

To **limit** or at least to **lower** the financial risk.

Depending on your attitude to risk and the information available to you, you can choose to set up

- a perfect hedge
- a delta hedge
- a gamma hedge or
- use some other strategy . . .

Perfect hedge

Fix a T-claim X.

A self-financing portfolio h is a **perfect hedge** against X if

$$V_T^S(h) = X \quad P - a.s.$$

If you set up a perfect hedge you know exactly how much money you will make.

Problem: We have seen that the replicating portfolio requires rebalancing in continuous time $\Rightarrow$ high transactions costs!

Sensitivity measures

Let $P(t,s)$ denote the pricing function of a portfolio based on **one** underlying asset with price process S_t.

The Greeks:

$$\Delta = \frac{\partial P}{\partial s} \qquad \theta = \frac{\partial P}{\partial t}$$

$$\Gamma = \frac{\partial^2 P}{\partial s^2} \qquad \rho = \frac{\partial P}{\partial r}$$

$$\mathcal{V} = \frac{\partial P}{\partial \sigma}$$

$\mathcal{V} =$ "vega"

Δ, Γ reflects the portfolio's sensitivity to small changes in the price of the underlying (financial risk).

$\rho, \mathcal{V}$ reflects the portfolio's sensitivity to misspecifications of the model!

Delta hedging

Goal: Make the portfolio insensitive to small changes in the price of the underlying.

A portfolio with $\Delta = 0$ is said to be **delta neutral**

Idea: Add a derivative to the portfolio. Since the price of the derivative is perfectly correlated with the price of the underlying it should be possible to choose portfolio weights so as to make the modified portfolio delta neutral.

Definition:
A position in the derivative is a **delta hedge** for the portfolio if the modified portfolio (original portfolio + derivative) is delta neutral.

Notation:

$P(t, s)$ the pricing function of the portfolio

$F(t, s)$ the pricing function of the derivative

x the number of derivatives to add to the portfolio

The value process of the modified portfolio is

$$V(t, s) = P(t, s) + x \cdot F(t, s)$$

The modified portfolio is delta neutral if

$$\frac{\partial V}{\partial s} = 0$$

i.e.

$$\frac{\partial P}{\partial s} + x\frac{\partial F}{\partial s} = 0$$

Solve for x!

The solution is

$$x = -\frac{\Delta_P}{\Delta_F}$$

Example: Suppose that we have sold a derivative with price function $G(t, s)$ and that we wish to hedge it using the underlying itself. Then we have that

$$P(t, s) = -G(t, s)$$

and that

$$F(t, s) = s.$$

For x we obtain

$$x = \Delta_G.$$

The delta of the derivative gives us the number of underlying we have to buy to hedge the derivative!

Problem: The portfolio has to be rebalanced when the price of the underlying changes!

Prop: For a continuously rebalanced delta hedge in the underlying, the value of the underlying and of the bank account (used to keep things self-financing) will replicate the derivative!

This means that the continuously rebalanced delta hedge described above is a perfect hedge!

Gamma hedging

A delta hedge is rebalanced because S and with that also Δ is changed.

Γ is a measure of how sensitive Δ is to changes in S.

$$\Gamma = \frac{\partial^2 P}{\partial s^2} = \frac{\partial \Delta}{\partial s}$$

Goal: Make the portfolio both delta and gamma neutral.

Idea: Add **two** derivatives to the portfolio.

Notation:

$P(t,s)$ pricing function of the portfolio

$F(t,s)$ pricing function of derivative 1

$G(t,s)$ pricing function of derivative 2

x_F the number of derivatives of type 1 to add to the portfolio

x_G the number of derivatives of type 2 to add to the portfolio

The value process of the modified portfolio is

$$V(t,s) = P(t,s) + x_F \cdot F(t,s) + x_G \cdot G(t,s)$$

We want the following to hold

$$\frac{\partial V}{\partial s} = 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial s^2} = 0$$

This yields the following system of equations

$$\begin{cases} \Delta_P + x_F \Delta_F + x_G \Delta_G & = & 0 \\ \Gamma_P + x_F \Gamma_F + x_G \Gamma_G & = & 0 \end{cases}$$

Solve for x_F and x_G!

Note that for the underlying itself we have

$$\Delta_S = 1 \qquad \text{och} \qquad \Gamma_S = 0$$

If you choose $G(t, s) = s$ you will get a triangular system which is easy to solve

$$\begin{cases} x_F & = & -\dfrac{\Gamma_P}{\Gamma_F} \\ x_S & = & \dfrac{\Delta_F \Gamma_P}{\Gamma_F} - \Delta_P \end{cases}$$