Pricing interest rate options

Assume that under Q

$$dr_t = \{a(t) - b(t)r_t\}dt + \sigma(t)dV_t,$$

where a, b, and σ are deterministic functions.

Then

1. r is a normal process

2. $p(t,T) = \exp\{A(t,T) - B(t,T)r(t)\}$

3. The price of an option on an S-bond is given by

$$\Pi_t = E^Q\left[e^{-\int_t^T r(s)ds} g(r_T) \middle| \mathcal{F}_t \right]$$

where

$$g(r) = \max\{e^{\{A(T,S)-B(T,S)r\}} - K, 0\}$$

Π_0 can be computed by intergrating w.r.t. the two dimensional normal vector

$$\left(r(T), \int_0^T r(s)ds\right)$$

To see that it is a normal vector let

$$Z_t = \int_0^t r(s)ds$$

Then

$$d\begin{bmatrix} r_t \\ Z_t \end{bmatrix} = \left\{\begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} -a & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r_t \\ Z_t \end{bmatrix}\right\} dt + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} dV$$

This is a linear SDE with an explicit solution, see Exercise 2.4 or Section 5.3.

Using the T-forward measure we obtain

$$\Pi_0 = p(0,T)E^T\left[g(r_T)\right]$$

where $p(0,T)$ is observable on the market and the integral is now w.r.t. a one dimensional distribution.

We need the distribution of r_T under Q^T.

We know that the Girsanov kernel from Q to Q^T is the volatility of p^T.

The Itô formula applied to
$p(t,T) = e^{\{A(t,T)-B(t,T)r(t)\}}$ yields

$$dp^T(t) = r(t)p^T(t)dt \underbrace{-\sigma(t)B(t,T)}_{volatility} p^T(t)dV$$

Remark: The fact that the local rate of return is equal to r need not be computed.

The dynamics of r under Q^T are

$$dr = \left(a_t - b_t r_t - B_t^T \sigma_t^2\right) dt + \sigma_t dV_t^T$$

The explicit solution is

$$\begin{aligned} r(T) &= e^{H(0,T)} r_0 + \int_0^T e^{H(s,T)} \{a_s - \sigma_s^2 B_s^T\} ds \\ &\quad + \int_0^T \underbrace{e^{H(s,T)} \sigma_s}_{determ.} dV_s^T \end{aligned}$$

where $H(s,t) = \int_s^t b(u) du$

Thus r is a normal process under Q^T with

$$\begin{aligned} E^T[r_T] &= f(0,T) \\ Var(r_T) &= E^T\left[\left(\int_0^T e^{H(T,s)} \sigma_s dV_s^T\right)^2\right] \\ &= \int_0^T e^{2H(T,s)} \sigma^2(s) ds \end{aligned}$$