Problem 1

(a) The price of the claim is in general given by

$$\Pi(0; X) = \frac{1}{1 + r} E^Q [X],$$

where $r$ is the risk-free interest rate and $Q$ is the equivalent martingale (risk-neutral) measure. We know that in this type of model

$$Q(\omega_1) = \frac{1 + r - d}{u - d} \quad \text{and} \quad Q(\omega_2) = 1 - Q(\omega_2).$$

In our case

$$r = 0, \quad u = \frac{120}{100} = 1.2 \quad \text{and} \quad d = \frac{80}{100} = 0.8;$$

hence

$$Q(\omega_1) = \frac{1 - 0.8}{1.2 - 0.8} = \frac{1}{2} \quad \text{and} \quad Q(\omega_2) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Finally

$$\Pi(0; X) = \frac{1}{2} \cdot \max(120 - 90, 0) + \frac{1}{2} \cdot \max(80 - 90, 0) = 15.$$

(b) Let $S$ be the price process of a traded asset. Any equivalent martingale measure (EMM) $Q$ satisfies

$$S_0 = \frac{1}{1 + r} E^Q [S_1],$$

where $r$ is the risk-free rate. The traded assets in this case are the bank account with zero interest rate, and the stock. Hence, any EMM $Q$ in our model must satisfy

$$1 = Q(\omega_1) \cdot 1 + Q(\omega_2) \cdot 1 + Q(\omega_3) \cdot 1$$

$$100 = Q(\omega_1) \cdot 120 + Q(\omega_2) \cdot 100 + Q(\omega_3) \cdot 80$$

Form the first equation:

$$Q(\omega_2) = 1 - Q(\omega_1) - Q(\omega_3),$$
and inserting this in the second yields

\[ 100 = 120Q(\omega_1) + 100\left(1 - Q(\omega_2) - Q(\omega_3)\right) + 80Q(\omega_3), \]

or

\[ Q(\omega_1) = Q(\omega_3) =: q. \]

Hence, any EMM \( Q \) must be on the form

\[ Q = (Q(\omega_1), Q(\omega_2), Q(\omega_3)) = (q, 1 - 2q, q). \]

But \( Q \) must be equivalent to \( P \), which in this case means that we also must have

\[ Q(\omega_i) > 0, \ i = 1, 2, 3. \]

This condition is fulfilled if

\[ q > 0 \text{ and } 1 - 2q > 0 \iff 0 < q < \frac{1}{2}. \]

Hence

\[ Q = \{(q, 1 - 2q, q) \mid q \in (0, 1/2)\}. \]

(c) Every arbitrage free price of the \( T \)-claim \( X \) in this model is given by (with \( r = 0 \))

\[ \Pi(0; X) = E^Q [X] \]

for some \( Q \in Q \). We get

\[ \Pi(0+; X) = Q(\omega_1)X(\omega_1) + Q(\omega_2)X(\omega_2) + Q(\omega_3)X(\omega_3) \]
\[ = q \max(120 - 90, 0) + (1 - 2q) \max(100 - 90, 0) + q \max(80 - 90, 0) \]
\[ = 30q + (1 - 2q)10 + 0 \]
\[ = 10 + 10q \]

for some \( q \in (0, 1/2) \). Hence, the set of arbitrage free prices of this \( T \)-claim is given by \((10, 15)\).

**Problem 2**

(a) In this case, the dynamics of the stock under the EMM \( Q \) where the bank account is the numeraire, is given by

\[ dS(t) = (r - \delta)S(t)dt + \sigma S(t)dW^Q(t), \]

where \( W^Q \) is a \( Q \)-Wiener process. The solution to this SDE satisfies

\[ S(T) = se^{(r-\delta-\sigma^2/2)T+\sigma W^Q(T)}, \]

for some \( \sigma \).
hence

\[ \Pi(0; X) = e^{-rT} E^Q \left[ S^2(T) \right] \]
\[ = e^{-rT} E^Q \left[ s^2 e^{(2r-2\delta-\sigma^2)T+2\sigma W^Q(T)} \right] \]
\[ = \left\{ W^Q(T) \sim N(0, \sqrt{T}) \right\} \]
\[ = s^2 e^{(r-2\delta-\sigma^2)T} e^{\frac{(2\sigma)^2}{2}T} \]
\[ = s^2 e^{(r-2\delta+\sigma^2)T} \]

(b) We start by calculating the price at \( T/2 \). During the interval \( [T/2, T] \) there are no dividend payments, so the dynamics is given by

\[ dS(t) = rS(t)dt + \sigma S(t)dW^Q(t), \quad t \in [T/2, T], \]

where \( W^Q \) is a \( Q \)-Wiener process. Hence

\[ \Pi(T/2; X) = e^{-r(T-T/2)} E^Q \left[ S^2(T)|F_{T/2} \right] \]
\[ = S^2(T/2) e^{(r+\sigma^2)T/2} \]
\[ =: F^1(t, S(T/2)). \]

The stock price drops with \( \gamma S(t-) \) at time \( t = T/2 \). Let \( F^0 \) denote the price function on \( [0, T/2) \). The matching (continuity) condition is

\[ F^0(T/2, s) = F^1(T/2, s - \gamma s) = F^1(T/2, (1 - \gamma)s). \]

Hence

\[ F^0(T/2, s) = (1 - \gamma)^2 s^2 e^{(r+\sigma^2)T/2}. \]

Again, we use the fact that we have the same the dynamics as above on \( [0, T/2) \).

\[ \Pi(0; X) = e^{-rT} E^Q \left[ \Pi(T/2; X) \right] \]
\[ = e^{-rT} E^Q \left[ F^0(T/2, S(T/2)) \right] \]
\[ = e^{-rT} E^Q \left[ (1 - \gamma)^2 S^2(T/2)e^{(r+\sigma^2)T/2} \right] \]
\[ = (1 - \gamma)^2 s^2 \sigma^2 T \left[ s^2 e^{(r-\sigma^2/2)T+2\sigma W^Q(T/2)} \right] \]
\[ = (1 - \gamma)^2 s^2 e^T e^{\frac{(2\sigma)^2}{2}T} \]
\[ = (1 - \gamma)^2 s^2 e^{(r+\sigma^2)T}. \]

Problem 3
We assume $a \neq 0$.

(a) We have a model with an affine term structure with 
\[ \alpha(t) = -a, \quad \beta(t) = 0, \quad \gamma(t) = 0 \text{ and } \delta(t) = \sigma^2. \]
Hence, the zero coupon bond prices are given by 
\[ p(t,T) = e^{A(t,T) - r(t)B(t,T)}, \]
where $A$ and $B$ solves 
\[ \frac{\partial A(t,T)}{\partial t} = -\frac{\sigma^2}{2} B^2(t,T) \quad \text{with} \quad A(T,T) = 0 \]
and 
\[ \frac{\partial B(t,T)}{\partial t} - aB(t,T) = -1 \quad \text{with} \quad B(T,T) = 0. \]
Using 
\[ \frac{d}{dt} \left( e^{at} B(t,T) \right) = ae^{at} B(t,T) + e^{at} \frac{\partial B}{\partial t} = e^{at} \left( aB(t,T) + \frac{\partial B}{\partial t} \right) = -e^{at} \]
we get 
\[ B(t,T) = \frac{1}{a} \left( 1 - e^{-a(T-t)} \right). \]
It follows that 
\[ A(T,T) - A(t,T) = -\frac{\sigma^2}{2} \int_t^T B^2(u,T)du = -\frac{\sigma^2}{2a^2} \int_t^T \left( 1 - e^{-a(T-u)} \right)^2 du, \]
and from this 
\[ A(t,T) = \frac{\sigma^2}{2a^2} \left[ T - t - \frac{2}{a} \left( 1 - e^{-a(T-t)} \right) + \frac{1}{2a} \left( 1 - e^{-2a(T-t)} \right) \right]. \]
Finally 
\[ f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + r(t) \frac{\partial B(t,T)}{\partial T} = \frac{\sigma^2}{2a^2} \left[ 1 + 2e^{-a(T-t)} - e^{-2a(T-t)} \right] + r(t)e^{-a(T-t)} = -\frac{\sigma^2}{2a^2} \left( 1 - e^{-a(T-t)} \right)^2 + r(t)e^{-a(T-t)}. \]

(b) We use the $Q^T$-forward measure: 
\[ \Pi(t;X) = p(t,T)E^{Q^T} \left[ X | \mathcal{F}_t \right]. \]
The dynamics of $r$ is given by
\[
    dr(t) = -ar(t)dt + \sigma dW^Q(t)
\]
\[
    = -ar(t)dt + \sigma \left( v(t, T)dt + dW^Q_T(t) \right)
\]
\[
    = \left( -ar(t)dt + \sigma v(t, T) \right)dt + \sigma dW^Q(t).
\]
where $v(t, T)$ is the volatility of the $T$-bond and $W^Q_T$ is a $Q^T$-Wiener process. Using Ito’s formula on $p(t, T) = e^{A(t,T) - B(t,T)r(t)}$ we get
\[
dp(t, T) = r(t)p(t, T)dt + [-\sigma B(t, T)]p(t, T)dW^Q(t).
\]
Hence $v(t, T) = -\sigma B(t, T)$ and
\[
    dr(t) = -\left( ar(t) + \frac{\sigma^2}{a} \left( 1 - e^{-a(T-t)} \right) \right)dt + \sigma dW^Q_T(t).
\]
Now,
\[
d(e^{at}r(t)) = ae^{at}r(t)dt + e^{at}dr(t) + ae^{at}dt dr(t) = 0
\]
\[
    = e^{at} (ar(t)dt + dr(t))
\]
\[
    = e^{at} \left( -b(t)dt + \sigma dW^Q_T(t) \right),
\]
and from this
\[
r(T) = r(t)e^{-a(T-t)} - \int_t^T e^{-a(T-u)}b(u)du + \sigma \int_t^T e^{-a(T-u)}dW^Q_T(u).
\]
Hence, $r(T)|\mathcal{F}_t$ is normally distributed. We know that
\[
    E^{Q_T}[r(T)|\mathcal{F}_t] = f(t, T)
\]
and the variance is given by
\[
    \text{Var}^{Q_T}(r(T)|\mathcal{F}_t) = \sigma^2 \int_t^T e^{-2a(T-u)}du = \frac{\sigma^2}{2a^2} \left( 1 - e^{-2a(T-t)} \right).
\]
It follows that
\[
    \Pi(t; X) = p(t, T)E^{Q_T}[e^{r(T)}|\mathcal{F}_t]
\]
\[
    = p(t, T)e^{E^{Q_T}[r(T)|\mathcal{F}_t] + \frac{1}{2} \text{Var}^{Q_T}(r(T)|\mathcal{F}_t)}
\]
\[
    = p(t, T)e^f(t, T) + \frac{\sigma^2}{2a^2} (1 - e^{-2a(T-t)}),
\]
with $p(t, T)$ and $f(t, T)$ from (a):
\[
p(t, T) = e^{\frac{\sigma^2}{2a^2} \left( T-t - \frac{1}{2} + (1-e^{-a(T-t)}) + \frac{a}{2} (1-e^{-2a(T-t)}) \right) - r(t) \frac{1-e^{-a(T-t)}}{a}}
\]
\[
f(t, T) = -\frac{\sigma^2}{2a^2} \left( 1 - e^{-a(T-t)} \right)^2 + r(t)e^{-a(T-t)}.
\]
Remark. When \( a = 0 \) we get

(a) Forward rate

\[ f(t, T) = r(t) + \frac{\sigma^2}{2}(T - t)^2. \]

(b) The price

\[ \Pi(t; X) = p(t, T)E^{Q_T} \left[ e^{r(T)} \mid \mathcal{F}_t \right] = e^{-\frac{\sigma^2}{2}(T-t)^3 - r(t)(T-t) + \frac{\sigma^2}{2}(T-t)^2 + \frac{\sigma^2}{2}(T-t)} \]

Problem 4

Let \( Q^T \) denote the \( T \)-forward measure for \( T > 0 \).

(a) We get

\[
E^{Q^T} \left[ r(T) \mid \mathcal{F}_t \right] = \frac{E^{Q} \left[ L^T(T)r(T) \mid \mathcal{F}_t \right]}{E^{Q} \left[ L^T(T) \mid \mathcal{F}_t \right]}
\]

\[
= E^{Q} \left[ L^T(T) \frac{r(T)}{L^T(t)} \mid \mathcal{F}_t \right]
\]

\[
= E^{Q} \left[ e^{-\int_t^T r(u)du} \frac{r(T)}{p(t, T)} \mid \mathcal{F}_t \right]
\]

\[
= \frac{1}{p(t, T)} E^{Q} \left[ \frac{\partial}{\partial T} \left( e^{-\int_t^T r(u)du} \right) \mid \mathcal{F}_t \right]
\]

\[
= -\frac{1}{p(t, T)} \frac{\partial}{\partial T} E^{Q} \left[ e^{-\int_t^T r(u)du} \mid \mathcal{F}_t \right]
\]

\[
= -\frac{1}{p(t, T)} \frac{\partial p(t, T)}{\partial T}
\]

\[
= f(t, T).
\]

(b) We have

\[
L(t; S, T) = \frac{1}{T - S} \left[ \frac{p(t, S)}{p(t, T)} - 1 \right].
\]

We know that under the forward measure \( Q^T \) the discounted price process \( p(t, S)/p(t, T) \) is a martingale. Since the constant \(-1\) is a martingale we have that \( p(t, S)/p(t, T) - 1 \) is also a martingale. Multiplying any martingale with a real number (in this case \( 1/(T - S) \)) we still have a martingale.

(c) We know that \( f(t; T, X) \) solves the equation

\[
\Pi(t; X - f(t; T, X)) = 0 \iff \Pi(t; X) = \Pi(t; f(t; T, X)) = p(t, T)f(t; T, X).
\]
Using the forward measure $Q^T$ we get

$$f(t; T) = \frac{\Pi(t; X)}{p(t, T)} = \frac{p(t, T)E^{Q^T}[X|\mathcal{F}_t]}{p(t, T)} = E^{Q^T}[X|\mathcal{F}_t].$$

**Problem 5**

The dynamics of $S_1$ and $S_2$ under the EMM $Q$ where $B$ is used as numeraire is given by

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1^Q(t) \text{ with } S_1(0) = s_1 > 0$$
$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2^Q(t) \text{ with } S_2(0) = s_2 > 0,$$

where $(W_1^Q, W_2^Q)$ is a two-dimensional $Q$-Wiener process. The solutions to these SDE’s are

$$S_i(T) = S_i(t)e^{(r-\sigma_i^2/2)(T-t)+\sigma_i(W_i^Q(T)-W_i^Q(t))}, \quad i = 1, 2.$$

(a) To determine the arbitrage free price at time $t \in [0, T]$ of the $T$-claim we start by noting that

$$X = \ln \left( \frac{S_1(T)}{S_2(T)} \right) = \ln S_1(T) - \ln S_2(T).$$

Hence

$$\Pi(t; X) = e^{-r(T-t)}E^Q[X|\mathcal{F}_t]$$
$$= e^{-r(T-t)}E^Q[\ln S_1(T) - \ln S_2(T)|\mathcal{F}_t]$$
$$= e^{-r(T-t)}E^Q[\ln S_1(t) + (r - \sigma_1^2/2)(T-t) + \sigma_1 \left( W_1^Q(T) - W_1^Q(t) \right)$$
$$- \ln S_2(t) - (r - \sigma_2^2/2)(T-t) - \sigma_2 \left( W_2^Q(T) - W_2^Q(t) \right) |\mathcal{F}_t]$$
$$= e^{-r(T-t)} \left( \ln \left( \frac{S_1(t)}{S_2(t)} \right) + (\sigma_2^2 - \sigma_1^2) \frac{T-t}{2} \right).$$

(b) To find the hedging portfolio $(h^B(t), h_1(t), h_2(t))$, we start with using the result that

$$h_i(t) = \frac{\partial F(t; S_1(t), S_2(t))}{\partial x_i}, \quad i = 1, 2,$$

where

$$F(t, x_1, x_2) = e^{-r(T-t)}E^Q_{x_1,x_2} \left[ \left( \frac{S_1(T)}{S_2(T)} \right) \right].$$

Hence,

$$h_1(t) = e^{-r(T-t)} \cdot \frac{1}{S_1(t)}$$
$$h_2(t) = -e^{-r(T-t)} \cdot \frac{1}{S_2(t)}$$
Finally,

\[
\begin{align*}
    h^B(t) &= \frac{F(t, S_1(t), S_2(t)) - (h_1(t)S_1(t) + h_2(t)S_2(t))}{B(t)} \\
    &= \frac{e^{-r(T-t)} \left( \ln \left( \frac{S_1(t)}{S_2(t)} \right) + (\sigma_2^2 - \sigma_1^2) \frac{T-t}{2} \right) - (e^{-r(T-t)} - e^{-r(T-t)})}{e^{rt}} \\
    &= e^{-rT} \left( \ln \left( \frac{S_1(t)}{S_2(t)} \right) + (\sigma_2^2 - \sigma_1^2) \frac{T-t}{2} \right).
\end{align*}
\]