EXAMINATION IN SF2975 FINANCIAL DERIVATIVES

Date: 2014-05-21, 14:00-19:00

Suggested solution

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Problem 1

(a) The price today of a simple claim $X = \Phi(S(T))$ with maturity at T is given by

$$\Pi(0,T) = \frac{1}{(1+r)^T} \sum_{k=0}^{T} \binom{T}{k} q_u^k q_d^{T-k} \Phi(S_0 u^k d^{T-u}),$$

where q_u and q_d are the risk-neutral probabilities, u and d is the size of an up and down move respectively and r is the risk-free rate. Here we have

$$r = 0 \text{ and } q_u = \frac{(1+r)-d}{u-d} = \frac{1-0.8}{1.2-0.8} = \frac{1}{2}$$

and $\Phi(x) = \max(50-x, 0)$. There are five possible values for the stock prices at time $T = 4$, but only in the case when we have four down moves does the option have a non-zero value. Hence we get

$$\Pi(0,4) = q_d^4 \cdot (50 - 100 \cdot d^4) = \left(\frac{1}{2}\right)^4 \cdot (50 - 100 \cdot 0.8^4) = \frac{9,04}{16} = 0.565.$$

(b) See the book.

Problem 2

The dynamcis of S under the unique EMM where the bank account is the numeraire is given by

$$dS(t) = (r-\delta)S(t)dt + \sigma S(t) dW^Q(t),$$

which has solution

$$S(T) = S(t) e^{(r-\delta-\sigma^2/2)\cdot(T-t) + \sigma(W^Q(T) - W^Q(t))}.$$

It follows that

$$\ln S(T) = \ln S(t) + (r - \delta - \sigma^2/2) \cdot (T-t) + \sigma(W^Q(T) - W^Q(t)).$$

The risk-neutral valuation formula gives the price at time t:

$$\Pi(t; X) = e^{-r(T-t)} E^Q \left[X | \mathcal{F}_t \right].$$

Now,

$$\begin{aligned}
X &= \mathbf{1}\left(\ln\left(S^{\beta}(T)\right) \geq K\right) \\
&= \mathbf{1}\left(\beta \ln(S(T)) \geq K\right) \\
&= \mathbf{1}\left(\ln S(t) + (r - \delta - \sigma^2/2) \cdot (T-t) + \sigma(W^Q(T) - W^Q(t)) \geq K/\beta\right) \\
&= \mathbf{1}\left(W^Q(T) - W^Q(t) \geq \frac{K/\beta - \ln S(t) - (r - \delta - \sigma^2/2) \cdot (T-t)}{\sigma}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Pi(t;X) &= e^{-r(T-t)} E^Q\left[\mathbf{1}\left(W^Q(T) - W^Q(t) \geq \frac{K/\beta - \ln S(t) - (r - \delta - \sigma^2/2) \cdot (T-t)}{\sigma}\right)\middle| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E^Q\left[\mathbf{1}\left(\frac{W^Q(T) - W^Q(t)}{\sqrt{T-t}} \geq \frac{K/\beta - \ln S(t) - (r - \delta - \sigma^2/2) \cdot (T-t)}{\sigma\sqrt{T-t}}\right)\middle| \mathcal{F}_t\right] \\
&= \left\{\frac{W^Q(T) - W^Q(t)}{\sqrt{T-t}}\middle| \mathcal{F}_t \sim N(0,1)\right\} \\
&= e^{-r(T-t)}\left(1 - N\left(\frac{K/\beta - \ln S(t) - (r - \delta - \sigma^2/2) \cdot (T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= e^{-r(T-t)} N\left(\frac{\ln S(t) + (r - \delta - \sigma^2/2) \cdot (T-t) - K/\beta}{\sigma\sqrt{T-t}}\right),
\end{aligned}$$

where N is the distribution function of a $N(0,1)$-distributed random variable.

Problem 3

(a) This is a model with an affine term structure. The zero-coupon bond price $p(t,T)$ is given by

$$p(t,T) = e^{A(t,T) - r(t)B(t,T)},$$

where A and B solves

$$\frac{\partial A}{\partial t}(t,T) = \theta(t)B(t,T) - \frac{\sigma^2}{2}B^2(t,T) \text{ with } A(T,T) = 0$$

and

$$\frac{\partial B}{\partial t} = -1 \text{ with } B(T,T) = 0.$$

We get

$$B(t,T) = T - t$$

from the last equation, and inserting this in the first equation yields

$$\frac{\partial A}{\partial t}(t,T) = \theta(t)(T-t) - \frac{\sigma^2}{2}(T-t)^2.$$

It follows that

$$A(T,T) - A(t,T) = \int_t^T \theta(s)(T-s)ds - \frac{\sigma^2}{2}\int_t^T (T-s)^2 ds,$$

and from this

$$-A(t,T) = \int_t^T \theta(s)(T-s)ds - \frac{\sigma^2}{6}(T-t)^3,$$

or

$$A(t,T) = \int_t^T \theta(s)(s-T)ds + \frac{\sigma^2}{6}(T-t)^3.$$

Hence

$$p(t,T) = e^{\int_t^T \theta(s)(s-T)ds + \frac{\sigma^2}{6}(T-t)^3 - r(t)(T-t)}.$$

(b) The theoretical instantaneous forward rates are given by

$$f(0,t) = -\frac{\partial}{\partial T}\ln p(0,t) = r(0)\frac{\partial B}{\partial T}(0,t) - \frac{\partial A}{\partial T}(0,t),\ t \geq 0.$$

With A and B from above, we get

$$f(0,t) = r(0) + \int_0^t \theta(s)ds - \frac{\sigma^2}{2}t^2$$

We want to match these theoretical forward rates with the observed:

$$f(0,t) = f^*(0,t)$$

$$\Leftrightarrow$$

$$r(0) + \int_0^t \theta(s)ds - \frac{\sigma^2}{2}t^2 = f^*(0,t)$$

Now we differentiate with respect to t, which yields

$$\theta(t) - \sigma^2 t = \frac{\partial f^*}{\partial T}(0,t).$$

It follows that in order for the model to be calibrated we should choose

$$\theta(t) = \frac{\partial f^*}{\partial T}(0,t) + \sigma^2 t,\ t \geq 0.$$

Problem 4

(a) The foreign bank account denoted in the domestic currency has value

$$\tilde{B}_f(t) = B_f(t)X(t)$$

and is the price of a traded asset in the domestic economy. Hence $\tilde{B}_f(t)/S(t)$ must be a Q^S-martingale. We have

$$d(\tilde{B}_f(t)/S(t)) = \tilde{B}_f(t)d(1/S(t)) + d\tilde{B}_f(t)/S(t) + d\tilde{B}_f(t)d(1/S(t)).$$

Now

$$\begin{aligned}
d(\tilde{B}_f(t)) &= d(B_f(t)X(t)) \\
&= B_f(t)dX(t) + X(t)dB_f(t) + dB_f(t)dX(t) \\
&= B_f(t)(\alpha_X X(t)dt + \sigma_X X(t)dW(t)) + X(t)r_f B_f(t)dt \\
&\quad + r_f B_f(t)dt(\alpha_X X(t)dt + \sigma_X X(t)dW(t)) \\
&= (r_f + \alpha_X)\tilde{B}_f(t)dt + \sigma_X \tilde{B}_f(t)dW(t)
\end{aligned}$$

and

$$\begin{aligned}
d(1/S(t)) &= -(1/S^2(t))dS(t) + \frac{1}{2}(2/S^3(t))(dS(t))^2 \\
&= -(1/S(t))(\alpha dt + \sigma dW(t)) + (1/S(t))\sigma^2 dt \\
&= (1/S(t))(\sigma^2 - \alpha)dt - \sigma dW(t)).
\end{aligned}$$

We get

$$\begin{aligned}
d(\tilde{B}_f(t)/S(t)) &= \tilde{B}_f(t)((1/S(t))(\sigma^2 - \alpha)dt - \sigma dW(t)) \\
&\quad + (1/S(t))(r_f + \alpha_X)\tilde{B}_f(t)dt + \sigma_X \tilde{B}_f(t)dW(t) \\
&\quad + \Big((r_f + \alpha_X)\tilde{B}_f(t)dt + \sigma_X \tilde{B}_f(t)dW(t)\Big)\Big(1/S(t))(\sigma^2 - \alpha)dt - \sigma dW(t)\Big) \\
&= (\tilde{B}_f(t)/S(t))\Big((\sigma^2 - \alpha + r_f + \alpha_X - \sigma\sigma_X)dt + (\sigma_X - \sigma)dW(t)\Big) \\
&= (\tilde{B}_f(t)/S(t))(\sigma_X - \sigma)\Big(\frac{\sigma^2 - \alpha + r_f + \alpha_X - \sigma\sigma_X}{\sigma_X - \sigma}dt + dW(t)\Big).
\end{aligned}$$

Hence,

$$dW^S(t) := \frac{\sigma^2 - \alpha + r_f + \alpha_X - \sigma\sigma_X}{\sigma_X - \sigma}dt + dW(t)$$

is a Q^S-Wiener process. Inserting this expression in the dynamics for X yields

$$\begin{aligned}
dX(t) &= \Big(\alpha_X - \sigma_X \frac{\sigma^2 - \alpha + r_f + \alpha_X - \sigma\sigma_X}{\sigma_X - \sigma}\Big)X(t)dt + \sigma_X X(t)dW^S(t) \\
&= \Big(\sigma\sigma_X + \frac{\sigma_X(\alpha - r_f) - \alpha_X\sigma}{\sigma_X - \sigma}\Big)X(t)dt + \sigma_X X(t)dW^S(t).
\end{aligned}$$

(b) Let $h = (h^S, \tilde{h})$ denote a self-financing portfolio. The dynamics of its value process is given by

$$\begin{aligned}
dV^h(t) &= h^S(t)dS(t) + \tilde{h}(t)d\tilde{B}_f(t) \\
&= \Big(h^S(t)\alpha S(t) + \tilde{h}(t)(r_f + \alpha_X)\tilde{B}_f(t)\Big)dt \\
&\quad + \Big(h^S(t)\sigma S(t) + \tilde{h}(t)\sigma_X \tilde{B}_f(t)\Big)dW(t).
\end{aligned}$$

Now choose h^S and $\tilde{h}$ such that

$$\begin{aligned} h^S(t)\sigma S(t) + \tilde{h}(t)\sigma_X \tilde{B}_f(t) &= 0 \\ h^S(t)\alpha S(t) + \tilde{h}(t)(r_f + \alpha_X)\tilde{B}_f(t) &= r_d V^h(t) = r_d\Big(h^S(t)S(t) + \tilde{h}(t)\tilde{B}_f(t)\Big), \end{aligned}$$

where r_d is the domestic risk-free interest rate we are looking for. From the first equation we get

$$\tilde{h}(t)\tilde{B}_f(t) = -\frac{\sigma}{\sigma_X} h^S S(t),$$

and inserting this in the second yields

$$\alpha h^S(t)S(t) - (r_f + \alpha_X)\frac{\sigma}{\sigma_X} h^S(t)S(t) = r_d\Big(h^S(t)S(t) - \frac{\sigma}{\sigma_X} h^S(t)S(t)\Big),$$

or

$$h^S(t)S(t)\left[\alpha - (r_f + \alpha_X)\frac{\sigma}{\sigma_X} - r_d + r_d\frac{\sigma}{\sigma_X}\right] = 0.$$

If follows that the bracket must be equal to zero, from which it follows that

$$r_d = \frac{\alpha\sigma_X - (r_f + \alpha_X)\sigma}{\sigma_X - \sigma}.$$

Problem 5

(a) The value at time $t \in [0, T]$ of the T-claim X is given by

$$\begin{aligned} \Pi(t; X) &= e^{-r(T-t)} E^Q\left[\sum_{i=1}^n \sqrt{S_i(T)}\,\middle|\, \mathcal{F}_t\right] \\ &= e^{-r(T-t)} \sum_{i=1}^n E^Q\left[S_i^{1/2}(T)\,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

For each $i = 1, \ldots, n$ we have the Q-dynamics

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i(t) dW_i^Q(t).$$

It follows that

$$S_i(T) = S_i(t) e^{(r - \sigma_i^2/2)\cdot(T-t) + \sigma_i(W^Q(T) - W^Q(t))},$$

and

$$S_i^{1/2}(T) = S_i(t)^{1/2}(t) e^{(r/2 - \sigma_i^2/4)\cdot(T-t) + (\sigma_i/2)\cdot(W^Q(T) - W^Q(t))}.$$

Hence

$$\begin{aligned}
\Pi(t;X) &= e^{-r(T-t)}\sum_{i=1}^{n} E^Q\left[S_i^{1/2}(t)e^{(r/2-\sigma_i^2/4)\cdot(T-t)+(\sigma_i/2)\cdot(W^Q(T)-W^Q(t))}\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\sum_{i=1}^{n} S_i^{1/2}(t)e^{(r/2-\sigma_i^2/4)\cdot(T-t)}E^Q\left[e^{(\sigma_i/2)\cdot(W_i^Q(T)-W_i^Q(t))}\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\sum_{i=1}^{n} S_i^{1/2}(t)e^{(r/2-\sigma_i^2/4)\cdot(T-t)}e^{(\sigma_i^2/8)\cdot(T-t)} \\
&= e^{-(r/2)\cdot(T-t)}\sum_{i=1}^{n} S_i^{1/2}(t)e^{-(\sigma_i^2/8)\cdot(T-t)}.
\end{aligned}$$

(b) Let $h=(h^B,h_1,\ldots,h_n)$ denote the hedging portfolio. Then

$$h_i(t)=\frac{\partial F(t,S_1(t),\ldots,S_n(t))}{\partial x_i},\ i=1,\ldots,n,$$

where

$$\begin{aligned}
F(t,x_1,\ldots,x_n) &= e^{-r(T-t)}E^Q_{t,x_1,\ldots,x_n}\left[\sum_{i=1}^{n}\sqrt{S_i(T)}\right] \\
&= e^{-(r/2)\cdot(T-t)}\sum_{i=1}^{n} S_i^{1/2}(t)e^{-(\sigma_i^2/8)\cdot(T-t)}.
\end{aligned}$$

Hence,

$$h_i(t)=\frac{e^{-(r/2+\sigma_i^2/8)\cdot(T-t)}S_i^{-1/2}(t)}{2},\ i=1,\ldots,n.$$

Finally, $h^B(t)$ is given by

$$\begin{aligned}
h^B(t) &= \frac{F(t,S_1(t),\ldots,S_n(t))-\sum_{i=1}^{n}h_i(t)S_i(t)}{B(t)} \\
&= \frac{1}{2}e^{-(3r/2)\cdot(T-t)}\sum_{i=1}^{n} S_i^{1/2}(t)e^{-(\sigma_i^2/8)\cdot(T-t)}.
\end{aligned}$$