Problem 1

(a) The price today of a simple claim $X = \Phi(S(T))$ with maturity at $T$ is given by

$$\Pi(0,T) = \frac{1}{(1+r)^T} \sum_{k=0}^T \binom{T}{k} q_u^k q_d^{T-k} \Phi(S_0 u^k d^{T-k}),$$

where $q_u$ and $q_d$ are the risk-neutral probabilities, $u$ and $d$ is the size of an up and down move respectively and $r$ is the risk-free rate. Here we have

$$r = 0 \quad \text{and} \quad q_u = \frac{(1+r) - d}{u - d} = \frac{1 - 0.8}{1.2 - 0.8} = \frac{1}{2}$$

and $\Phi(x) = \max(50 - x, 0)$. There are five possible values for the stock prices at time $T = 4$, but only in the case when we have four down moves does the option have a non-zero value. Hence we get

$$\Pi(0,4) = q_d^4 \cdot (50 - 100 \cdot d^4) = \left(\frac{1}{2}\right)^4 \cdot (50 - 100 \cdot 0.8^4) = \frac{9.04}{16} = 0.565.$$  

(b) See the book.

Problem 2

The dynamics of $S$ under the unique EMM where the bank account is the numeraire is given by

$$dS(t) = (r - \delta)S(t)dt + \sigma S(t)dW^Q(t),$$

which has solution

$$S(T) = S(t)e^{(r-\delta-\sigma^2/2) \cdot (T-t) + \sigma (W^Q(T) - W^Q(t))}.$$  

It follows that

$$\ln S(T) = \ln S(t) + (r - \delta - \sigma^2/2) \cdot (T - t) + \sigma (W^Q(T) - W^Q(t)).$$

The risk-neutral valuation formula gives the price at time $t$:

$$\Pi(t; X) = e^{-r(T-t)} E^Q [X | \mathcal{F}_t].$$
Now,

\[ X = 1 \left( \ln \left( S^\beta(T) \right) \geq K \right) \]

\[ = 1 \left( \beta \ln(S(T)) \geq K \right) \]

\[ = 1 \left( \ln S(t) + (r - \delta - \sigma^2/2) \cdot (T - t) + \sigma(W^Q(T) - W^Q(t)) \geq K/\beta \right) \]

\[ = 1 \left( W^Q(T) - W^Q(t) \geq \frac{K/\beta - \ln S(t) - (r - \delta - \sigma^2/2) \cdot (T - t)}{\sigma} \right). \]

Hence,

\[
\Pi(t; X) = e^{-r(T-t)}E^Q \left[ 1 \left( W^Q(T) - W^Q(t) \geq \frac{K/\beta - \ln S(t) - (r - \delta - \sigma^2/2) \cdot (T - t)}{\sigma} \right) \bigg| \mathcal{F}_t \right]
\]

\[
= e^{-r(T-t)}E^Q \left[ \left\{ W^Q(T) - W^Q(t) \bigg| \mathcal{F}_t \bigg| \mathcal{N}(0,1) \right\} \right]
\]

\[
= e^{-r(T-t)} \left( 1 - N \left( \frac{K/\beta - \ln S(t) - (r - \delta - \sigma^2/2) \cdot (T - t)}{\sigma \sqrt{T-t}} \right) \right)
\]

where \( N \) is the distribution function of a \( \mathcal{N}(0,1) \)-distributed random variable.

**Problem 3**

(a) This is a model with an affine term structure. The zero-coupon bond price \( p(t, T) \) is given by

\[ p(t, T) = e^{A(t, T) - r(t)B(t, T)}, \]

where \( A \) and \( B \) solves

\[ \frac{\partial A}{\partial t}(t, T) = \theta(t)B(t, T) - \frac{\sigma^2}{2}B^2(t, T) \text{ with } A(T, T) = 0 \]

and

\[ \frac{\partial B}{\partial t} = -1 \text{ with } B(T, T) = 0. \]

We get

\[ B(t, T) = T - t \]

from the last equation, and inserting this in the first equation yields

\[ \frac{\partial A}{\partial t}(t, T) = \theta(t)(T - t) - \frac{\sigma^2}{2}(T - t)^2. \]

It follows that

\[ A(T, T) - A(t, T) = \int_t^T \theta(s)(T - s)ds - \frac{\sigma^2}{2} \int_t^T (T - s)^2ds, \]
and from this

\[-A(t, T) = \int_t^T \theta(s)(T - s)ds - \frac{\sigma^2}{6}(T - t)^3,\]

or

\[A(t, T) = \int_t^T \theta(s)(s - T)ds + \frac{\sigma^2}{6}(T - t)^3.\]

Hence

\[p(t, T) = e^{\int_t^T \theta(s)(s - T)ds} + \frac{\sigma^2}{2}(T - t)^3 - r(T - t).\]

(b) The theoretical instantaneous forward rates are given by

\[f(0, t) = -\frac{\partial}{\partial T} \ln p(0, t) = r(0) \frac{\partial B}{\partial T}(0, t) - \frac{\partial A}{\partial T}(0, t), \ t \geq 0.\]

With A and B from above, we get

\[f(0, t) = r(0) + \int_0^t \theta(s)ds - \frac{\sigma^2}{2} t^2\]

We want to match these theoretical forward rates with the observed:

\[f(0, t) = f^*(0, t)\]

\[\Leftrightarrow\]

\[r(0) + \int_0^t \theta(s)ds - \frac{\sigma^2}{2} t^2 = f^*(0, t)\]

Now we differentiate with respect to \(t\), which yields

\[\theta(t) - \sigma^2 t = \frac{\partial f^*}{\partial T}(0, t).\]

It follows that in order for the model to be calibrated we should choose

\[\theta(t) = \frac{\partial f^*}{\partial T}(0, t) + \sigma^2 t, \ t \geq 0.\]

Problem 4

(a) The foreign bank account denoted in the domestic currency has value

\[\tilde{B}_f(t) = B_f(t)X(t)\]

and is the price of a traded asset in the domestic economy. Hence \(\tilde{B}_f(t)/S(t)\) must be a \(Q^S\)-martingale. We have

\[d(\tilde{B}_f(t)/S(t)) = \tilde{B}_f(t)d(1/S(t)) + d\tilde{B}_f(t)/S(t) + d\tilde{B}_f(t)d(1/S(t)).\]

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Now
\[
d(\bar{B}_f(t)) = d(B_f(t)X(t)) \\
= B_f(t)dX(t) + X(t)dB_f(t) + dB_f(t)dX(t) \\
= B_f(t)(\alpha_X X(t)dt + \sigma_X X(t)dW(t)) + X(t)r_f dB_f(t)dt \\
+ r_f dB_f(t)dt(\alpha_X X(t)dt + \sigma_X X(t)dW(t)) \\
= (r_f + \alpha_X)\bar{B}_f(t)dt + \sigma_X \bar{B}_f(t)dW(t)
\]

and
\[
d(1/S(t)) = -(1/S^2(t))dS(t) + \frac{1}{2}(2/S^3(t))(dS(t))^2 \\
= -(1/S(t))(\alpha dt + \sigma dW(t)) + (1/S(t))\sigma^2 dt \\
= (1/S(t))(\sigma^2 - \alpha)dt - \sigma dW(t).
\]

We get
\[
d(\bar{B}_f(t)/S(t)) = \bar{B}_f(t)((1/S(t))(\sigma^2 - \alpha)dt - \sigma dW(t)) \\
+ (1/S(t))(r_f + \alpha_X)\bar{B}_f(t)dt + \sigma_X \bar{B}_f(t)dW(t) \\
+ \left( (r_f + \alpha_X)\bar{B}_f(t)dt + \sigma_X \bar{B}_f(t)dW(t) \right) \left( 1/S(t)(\sigma^2 - \alpha)dt - \sigma dW(t) \right) \\
= (\bar{B}_f(t)/S(t))\left( (\sigma^2 - \alpha + r_f + \alpha_X - \sigma \sigma_X)dt + (\sigma_X - \sigma)dW(t) \right) \\
= (\bar{B}_f(t)/S(t))(\sigma_X - \sigma)\left( \frac{\sigma^2 - \alpha + r_f + \alpha_X - \sigma \sigma_X}{\sigma_X - \sigma} dt + dW(t) \right).
\]

Hence,
\[
dW^S(t) := \frac{\sigma^2 - \alpha + r_f + \alpha_X - \sigma \sigma_X}{\sigma_X - \sigma} dt + dW(t)
\]
is a $Q^S$-Wiener process. Inserting this expression in the dynamics for $X$ yields
\[
dx(t) = \left( \alpha_X - \sigma_X \frac{\sigma^2 - \alpha + r_f + \alpha_X - \sigma \sigma_X}{\sigma_X - \sigma} \right) X(t)dt + \sigma_X X(t)dW^S(t) \\
= \left( \sigma_X + \frac{\sigma_X(\alpha - r_f - \alpha_X \sigma)}{\sigma_X - \sigma} \right) X(t)dt + \sigma_X X(t)dW^S(t).
\]

(b) Let $h = (h^S, \tilde{h})$ denote a self-financing portfolio. The dynamics of its value process is given by
\[
dV^h(t) = h^S(t)dS(t) + \tilde{h}(t)d\bar{B}_f(t) \\
= \left( h^S(t)\alpha S(t) + \tilde{h}(t)(r_f + \alpha_X)\bar{B}_f(t) \right)dt \\
+ \left( h^S(t)\sigma S(t) + \tilde{h}(t)\sigma_X \bar{B}_f(t) \right)dW(t).
\]
Now choose $h^S$ and $\tilde{h}$ such that

\begin{align*}
    h^S(t)\sigma S(t) + \tilde{h}(t)\sigma_X \tilde{B}_f(t) &= 0 \\
    h^S(t)\alpha S(t) + \tilde{h}(t)(r_f + \alpha_X) \tilde{B}_f(t) &= r_d V^h(t) = r_d \left( h^S(t)S(t) + \tilde{h}(t)\tilde{B}_f(t) \right),
\end{align*}

where $r_d$ is the domestic risk-free interest rate we are looking for. From the first equation we get

\[ \tilde{h}(t)\tilde{B}_f(t) = -\frac{\sigma}{\sigma_X} h^S(t), \]

and inserting this in the second yields

\[ \alpha h^S(t)S(t) - (r_f + \alpha_X)\frac{\sigma}{\sigma_X} h^S(t)S(t) = r_d \left( h^S(t)S(t) - \frac{\sigma}{\sigma_X} h^S(t)S(t) \right), \]

or

\[ h^S(t)S(t) \left[ \alpha - (r_f + \alpha_X)\frac{\sigma}{\sigma_X} - r_d + r_d \frac{\sigma}{\sigma_X} \right] = 0. \]

If follows that the bracket must be equal to zero, from which it follows that

\[ r_d = \frac{\alpha \sigma_X - (r_f + \alpha_X)\sigma}{\sigma_X - \sigma}. \]

**Problem 5**

(a) The value at time $t \in [0, T]$ of the $T$-claim $X$ is given by

\[ \Pi(t; X) = e^{-r(T-t)} E^Q \left[ \sum_{i=1}^n \sqrt{S_i(T)} \left| \mathcal{F}_t \right] \right] = e^{-r(T-t)} \sum_{i=1}^n E^Q \left[ S_i^{1/2}(T) \left| \mathcal{F}_t \right. \right]. \]

For each $i = 1, \ldots, n$ we have the $Q$-dynamics

\[ dS_i(t) = r S_i(t) dt + \sigma_i S_i(t) dW^Q_i(t). \]

It follows that

\[ S_i(T) = S_i(t)e^{(r - \sigma_i^2/2)(T-t) + \sigma_i (W^Q(T) - W^Q(t))}, \]

and

\[ S_i^{1/2}(T) = S_i(t)^{1/2}e^{(r/2 - \sigma_i^2/4)(T-t) + (\sigma_i/2)(W^Q(T) - W^Q(t))}. \]
Hence

$$
\Pi(t; \mathcal{X}) = e^{-r(T-t)} \sum_{i=1}^{n} E^{Q}\left[ S_{i}^{1/2}(t)e^{(r/2-\sigma_{i}^{2}/4):(T-t)+(\sigma_{i}/2):(W_{i}^{Q}(T)-W_{i}^{Q}(t))} \right] F_{t}
$$

$$
= e^{-r(T-t)} \sum_{i=1}^{n} S_{i}^{1/2}(t)e^{(r/2-\sigma_{i}^{2}/4):(T-t)} E^{Q}\left[ e^{(\sigma_{i}/2):(W_{i}^{Q}(T)-W_{i}^{Q}(t))} \right] F_{t}
$$

$$
= e^{-r(T-t)} \sum_{i=1}^{n} S_{i}^{1/2}(t)e^{(r/2-\sigma_{i}^{2}/4):(T-t)} e^{(\sigma_{i}^{2}/8):(T-t)}
$$

$$
= e^{-(r/2):(T-t)} \sum_{i=1}^{n} S_{i}^{1/2}(t)e^{-(\sigma_{i}^{2}/8):(T-t)}.
$$

(b) Let $h = (h^{B}, h_{1}, \ldots, h_{n})$ denote the hedging portfolio. Then

$$
h_{i}(t) = \frac{\partial F(t, S_{1}(t), \ldots, S_{n}(t))}{\partial x_{i}}, \quad i = 1, \ldots, n,
$$

where

$$
F(t, x_{1}, \ldots, x_{n}) = e^{-r(T-t)} E^{Q}_{t,x_{1},...,x_{n}} \left[ \sum_{i=1}^{n} \sqrt{S_{i}(T)} \right]
$$

$$
= e^{-(r/2):(T-t)} \sum_{i=1}^{n} S_{i}^{1/2}(t)e^{-(\sigma_{i}^{2}/8):(T-t)}.
$$

Hence,

$$
h_{i}(t) = \frac{e^{-(r/2+\sigma_{i}^{2}/8):(T-t)}S_{i}^{-1/2}(t)}{2}, \quad i = 1, \ldots, n.
$$

Finally, $h^{B}(t)$ is given by

$$
h^{B}(t) = \frac{F(t, S_{1}(t), \ldots, S_{n}(t)) - \sum_{i=1}^{n} h_{i}(t)S_{i}(t)}{B(t)}
$$

$$
= \frac{1}{2} e^{-(3r/2):(T-t)} \sum_{i=1}^{n} S_{i}^{1/2}(t)e^{-(\sigma_{i}^{2}/8):(T-t)}.
$$