Problem 1

(a) We use Feynman-Kac. It follows that

\[ F(t,x) = E_t,x \left[ e^{a(T-t)} X(T) \right], \]

where \( X \) has dynamics

\[ dX(t) = bX(t) dW(t), \]

and \( W \) is a 1-dimensional Wiener process. The solution to this SDE is

\[ X(T) = X(t) e^{-\frac{b^2}{2}(T-t)} + b(W(T) - W(t)), \]

and we get

\[ F(t,x) = e^{a(T-t)} x. \]

Alternatively we realise that \( X \) is a martingale, from which it follows that

\[ F(t,x) = e^{a(T-t)} E_t,x [X(T)] = e^{a(T-t)} x. \]

(b) We know that if \( f : [0, T] \to \mathbb{R} \) satisfies

\[ \int_0^T f^2(u) du < \infty, \]

then

\[ \int_0^T f(u) dW(u) \sim N \left( 0, \sqrt{\int_0^T f^2(u) du} \right). \]

In our case \( f(u) = u^2 \). Since

\[ \int_0^T (u^2)^2 du = \int_0^T u^4 du = \frac{T^5}{5} < \infty, \]

we get

\[ X = \int_0^T u^2 dW(u) \sim N \left( 0, \sqrt{\frac{T^5}{5}} \right). \]
(c) See the book.

**Problem 2**

(a) The risk-neutral valuation formula yields

\[
\Pi(t; X) = e^{-r(T-t)} E^Q \left[ (\ln(S(T)))^2 \right| F_t],
\]

where under \(Q\) the dynamics of \(S\) are given by

\[
dS(t) = rS(t)dt + \sigma S(t)dW^Q(t).
\]

Since

\[
S(T) = S(t)e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W^Q(T) - W^Q(t))},
\]

we get

\[
\ln S(T) - \ln S(t) + (r - \frac{\sigma^2}{2})(T-t) + \sigma(W^Q(T) - W^Q(t)) = 0.
\]

It follows that

\[
\ln S(T)|F_t \sim N \left( \ln S(t) + (r - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t} \right).
\]

Now

\[
E^Q \left[ (\ln S(T))^2 \right| F_t] = \text{Var}^Q (\ln S(T)|F_t) + (E^Q [\ln S(T)|F_t])^2
\]

\[
= \sigma^2(T-t) + (\ln S(t) + (r - \frac{\sigma^2}{2})(T-t))^2,
\]

and for \(t \in [0,T]\)

\[
\Pi(t; X) = e^{-r(T-t)} \left( \sigma^2(T-t) + (\ln S(t) + (r - \frac{\sigma^2}{2})(T-t))^2 \right).
\]

(b) We know that the hedging portfolio is given by

\[
h^S(t) = \frac{\partial F}{\partial s}(t, S(t)),
\]

where

\[
F(t, s) = E^Q_{t,s} \left[ (\ln S(T))^2 \right]
\]

\[
= e^{-r(T-t)} \left( \sigma^2(T-t) + (\ln s + (r - \frac{\sigma^2}{2})(T-t))^2 \right),
\]

and that

\[
h^B(t) = \frac{F(t, S(t)) - S(t) \frac{\partial F}{\partial s}(t, S(t))}{B(t)}.
\]
It follows that
\[ h_S(t) = \frac{2}{S(t)} e^{-r(T-t)} \left( \ln S(t) + (r - \sigma^2/2)(T-t) \right) \]
and
\[ h_B(t) = e^{-rT} \left[ 2(\sigma^2 - r)(T-t) + \left( \ln S(t) + (r - \sigma^2/2)(T-t) \right)^2 - 2 \ln S(t) \right]. \]

**Problem 3**

(a) The Q-dynamics of \( f(t, T) \) are given by
\[ df(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) \, du \, dt + \sigma(t, T) dW^Q(t), \]
where \( W^Q \) is a 1-dimensional Q-Wiener process. With
\[ \sigma(t, T) = \frac{\sigma_0}{1 + a(T-t)} \]
we get the following drift under \( Q \):
\[ \sigma(t, T) \int_t^T \sigma(t, u) \, du = \frac{\sigma_0}{1 + a(T-t)} \int_t^T \frac{\sigma_0}{1 + a(u-t)} \, du = \frac{\sigma_0}{1 + a(T-t)} \left[ \frac{\sigma_0}{a} \cdot \ln(1 + a(u-t)) \right]_t^T = \frac{\sigma_0^2}{a} \cdot \ln(1 + a(T-t)). \]
Hence
\[ df(t, T) = \frac{\sigma_0^2}{a} \cdot \ln(1 + a(T-t)) \, dt + \frac{\sigma_0}{1 + a(T-t)} dW^Q(t). \]
The initial value is the the same as under \( P \):
\[ f(0, T) = f^*(0, T). \]
Under \( Q \), the dynamics of \( p(t, T) \) is given by
\[ dp(t, T) = r(t) p(t, T) \, dt + \left( - \int_t^T \sigma(t, u) \, du \right) p(t, T) dW^Q(u). \]
In our case
\[ - \int_t^T \sigma(t, u) \, du = - \int_t^T \frac{\sigma_0}{1 + a(u-t)} \, du = - \frac{\sigma_0}{a} \ln(1 + a(T-t)), \]
so we get Q-dynamics
\[ dp(t, T) = r(t) p(t, T) \, dt - \frac{\sigma_0}{a} \ln(1 + a(T-t)) p(t, T) dW^Q(t) \]
with initial values
\[ p(0, T) = e^{-\int_0^T f^*(0, u) \, du}. \]
(b) See the book.

**Problem 4**

(a) We know that for \( t \in [0, T] \)

\[
\Pi(t; X) = p(t, T)E^{Q_T}[X|\mathcal{F}_t],
\]

where \( p(t, T) \) is the price at time \( t \) of a ZCB maturing at \( T \) and \( Q_T \) is the \( T \)-forward measure. The model

\[ dr(t) = \sigma_0 dW^Q(t) \]

has an ATS with, in the language of the course book,

\[ \alpha(t) = \beta(t) = \gamma(t) = 0 \text{ and } \delta(t) = \sigma_0^2. \]

Hence, the ZCB prices in this model are given by

\[ p(t, T) = e^{A(t, T) - B(t, T)r(t)}, \]

where \( A \) and \( B \) solves the system of equations

\[
\begin{cases}
\frac{\partial A(t, T)}{\partial t} = -\frac{\sigma_0^2}{2} B^2(t, T) \\
A(T, T) = 0
\end{cases}
\]

and

\[
\begin{cases}
\frac{\partial B(t, T)}{\partial t} = -1. \\
B(T, T) = 0
\end{cases}
\]

The solution to the last equation is given by

\[ B(t, T) = T - t, \]

and we then get

\[ A(T, T) - A(t, T) = -\int_t^T \frac{\sigma_0^2}{2} (T - u)^2 du = -\frac{\sigma_0^2}{2} \frac{(T-t)^3}{3}, \]

or

\[ A(t, T) = \frac{\sigma_0^2 (T - t)^3}{6}. \]

Hence

\[ p(t, T) = e^{\frac{\sigma_0^2 (T - t)^3}{6} - (T-t)r(t)}. \]

The forward rates are given by

\[ f(t, T) = -\frac{\partial \ln p(t, T)}{\partial T} = \frac{\partial}{\partial T} \left((T-t)r(t) - \frac{\sigma_0^2 (T - t)^3}{6}\right) = r(t) - \frac{\sigma_0^2 (T - t)^2}{2}. \]
(b) In general

\[ dr(t) = \mu(t, r(t)) dt + \sigma(t, r(t)) dW(t) \]
\[ = \mu_Q(t, r(t)) dt + \sigma(t, r(t)) dW^Q(t), \]

where \( W \) and \( W^Q \) is a Wiener process under \( P \) and \( Q \) respectively. We know that absence of arbitrage implies

\[ \mu_Q(t, r(t)) = \mu(t, r(t)) - \lambda(t, r(t)) \sigma(t, r(t)). \]

In our case we have

\[ \mu_Q(t, r(t)) = 0, \quad \lambda(t, r(t)) = \lambda_0 \quad \text{and} \quad \sigma(t, r(t)) = \sigma_0, \]

which implies

\[ \mu(t, r(t)) = \lambda_0 \sigma_0 \]

and

\[ dr(t) = \lambda_0 \sigma_0 dt + \sigma_0 dW(t). \]

Thus

\[ r(T) = r(0) + \lambda_0 \sigma_0 T + \sigma_0 W(T) \sim N \left( r(0) + \lambda_0 \sigma_0 T, \sigma_0 \sqrt{T} \right), \]

and we get

\[ P(r(T) < 0) = P(r(T) \leq 0) \]
\[ = P \left( \frac{r(T) - r(0) - \lambda_0 \sigma_0 T}{\sigma_0 \sqrt{T}} \leq \frac{-r(0) - \lambda_0 \sigma_0 T}{\sigma_0 \sqrt{T}} \right) \]
\[ = \Phi \left( -\frac{r(0) + \lambda_0 \sigma_0 T}{\sigma_0 \sqrt{T}} \right). \]

**Problem 5**

(a) We know that \( S_f(t)X(t)/B_d(t) \) should be a \( Q^d \)-martingale. First of all

\[ d(S_f(t)X(t)) = S_f(t) dX(t) + X(t) dS_f(t) + dS_f(t) dX(t) \]
\[ = \alpha_X S_f(t) X(t) dt + \sigma_X S_f(t) X(t) dW(t) + \alpha_f S_f(t) X(t) dt \]
\[ + \sigma_f S_f(t) X(t) dW(t) + \sigma_f \sigma_X S_f(t) X(t) dt \]
\[ = (\alpha_X + \alpha_f + \sigma_f \sigma_X) \frac{S_f(t) X(t)}{B_d(t)} dt + (\sigma_X + \sigma_f) S_f(t) X(t) dW(t). \]
We then get
\[
\begin{align*}
\frac{d}{B_d(t)} \left( \frac{S_f(t)X(t)}{B_d(t)} \right) &= d \left( e^{-rt} [S_f(t)X(t)] \right) \\
&= (\alpha_X + \alpha_f + \sigma_f \sigma_X - r_d) \frac{S_f(t)X(t)}{B_d(t)} dt \\
&\quad + (\sigma_X + \sigma_f) \frac{S_f(t)X(t)}{B_d(t)} dW(t) \\
&= (\alpha_X + \alpha_f + \sigma_f \sigma_X - r_d) \frac{S_f(t)X(t)}{B_d(t)} dt \\
&\quad + (\sigma_X + \sigma_f) \frac{S_f(t)X(t)}{B_d(t)} (dW^d(t) + \varphi(t)dt) \\
&= (\alpha_X + \alpha_f + \sigma_f \sigma_X - r_d + (\sigma_X + \sigma_f) \varphi(t)) \frac{S_f(t)X(t)}{B_d(t)} dt \\
&\quad + (\sigma_X + \sigma_f) \frac{S_f(t)X(t)}{B_d(t)} dW^d(t),
\end{align*}
\]

where \( W^d \) is a \( Q^d \)-Wiener process. By choosing
\[
\varphi(t) = -\frac{\alpha_X + \alpha_f + \sigma_f \sigma_X - r_d}{\sigma_X + \sigma_f}
\]
we see that the discounted price process is a \( Q^d \)-martingale. It follows that the dynamics of \( X \) under \( Q^d \) is
\[
\begin{align*}
\frac{dX}{dt} &= \alpha_X X(t) dt + \sigma_X X(t) \left( dW^d(t) - \frac{\alpha_X + \alpha_f + \sigma_f \sigma_X - r_d}{\sigma_X + \sigma_f} \right) \\
&= \left( \alpha_X - \sigma_X \frac{\alpha_X + \alpha_f + \sigma_f \sigma_X - r_d}{\sigma_X + \sigma_f} \right) X(t) dt + \sigma_X X(t) dW^d(t).
\end{align*}
\]

(b) The price at \( t \in [0, T] \) of the \( T \)-claim \( Z = S_f(T)X(T) \) is given by
\[
\Pi(t; Z) = e^{-r_d(T-t)} E^{Q^d} \left[ S_f(T)X(T) | \mathcal{F}_t \right]
\]
\[
= e^{r_d t} E^{Q^d} \left[ e^{-r_d T} S_f(T)X(T) | \mathcal{F}_t \right]
\]
\[
= \left\{ e^{-r_d t} S_f(t)X(t) \text{ is a } Q^d \text{-martingale} \right\}
\]
\[
= e^{r_d t} e^{-r_d t} S_f(t)X(t)
\]
\[
= S_f(t)X(t).
\]