EXAMINATION IN SF2975 FINANCIAL DERIVATIVES

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Suggested solutions.

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Problem 1

(a) The solution to the GBM SDE is in general

$$S(t) = S(u)e^{(\alpha-\sigma^2/2)(t-u)+\sigma(W(t)-W(u))}$$

for any $0 \le u \le t$. With $t = T$ and $u = T_1$ we get

$$S(T) = S(T_1)e^{(\alpha-\sigma^2/2)(T-T_1)+\sigma(W(T)-W(T_1))}.$$

Now the jump condition yields

$$S(T_1-) = S(T_1) + \delta S(T_1-) \quad \Leftrightarrow \quad S(T_1) = (1-\delta)S(T_1-).$$

Using again the solution of the GBM SDE, now with $t = T_1-$ and $u = 0$, we get

$$S(T_1-) = S(0)e^{(\alpha-\sigma^2/2)T_1+\sigma W(T_1)}.$$

Here we have used that both the function $t \mapsto t$ and the stochastic process $W(t)$ are continuous, so that we have $t- = t$ and $W(t-) = W(t)$ for every t. It follows that

$$\begin{aligned} S(T) &= \underbrace{(1-\delta)S(T_1-)}_{=S(T_1)}\, e^{(\alpha-\sigma^2/2)(T-T_1)+\sigma(W(T)-W(T_1))} \\ &= (1-\delta)S(0)e^{(\alpha-\sigma^2/2)T+\sigma W(T)}. \end{aligned}$$

Finally

$$E\left[S(T)\right] = (1-\delta)S(0)E\left[e^{(\alpha-\sigma^2/2)T+\sigma W(T)}\right] = (1-\delta)S(0)e^{\alpha T}.$$

(b) The solution to the SDE can be written

$$X(T) = X(t) + \int_t^T (a+bu)^2 dW(u).$$

We know that

$$\int_t^T (a+bu)^2 dW(u) \,|\, \mathcal{F}_t^W \sim N\left(0, \sqrt{\int_t^T (a+bu)^4 du}\right).$$

Now

$$\int_t^T (a+bu)^4 du = \frac{1}{b}\left[\frac{(a+bu)^5}{5}\right]_t^T = \frac{(a+bT)^5-(a+bt)^5}{5b}.$$

so we get

$$X(T) \,|\, \mathcal{F}_t^W \sim N\left(X(t), \sqrt{\frac{(a+bT)^5-(a+bt)^5}{5b}}\right).$$

(c) In a model with ATS we have

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)} =: f(r(t)).$$

Since

$$f''(x) = B(t,T)^2 e^{A(t,T)-B(t,T)x} > 0$$

we conclude that the ZCB price is convex as a function of the short rate.

Problem 2

Under Q, the dynamics of S is

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t),$$

where W^Q is a Q-Wiener process. The price at time $t \in [0, T_0]$ is given by

$$\begin{aligned}\Pi(t;X) &= e^{-r(T-t)}E^Q\left[\int_{T_0}^{T}(S(u)-S(T_0))du\,\middle|\,\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\int_{T_0}^{T}\left(E^Q\left[S(u)|\mathcal{F}_t\right]-E^Q\left[S(T_0)|\mathcal{F}_t\right]\right)du.\end{aligned}$$

Now

$$S(u) = S(t)e^{(r-\sigma^2/2)(u-t)+\sigma(W(u)-W(t))},$$

so

$$E^Q\left[S(u)|\mathcal{F}_t\right] = S(t)e^{r(u-t)}$$

and

$$E^Q\left[S(T_0)|\mathcal{F}_t\right] = S(t)e^{r(T_0-t)}.$$

It follows that

$$\begin{aligned}\Pi(t;X) &= e^{-r(T-t)}\int_{T_0}^{T}\left(S(t)e^{r(u-t)}-S(t)e^{r(T_0-t)}\right)du \\ &= e^{-r(T-t)}S(t)e^{-rt}\left(\int_{T_0}^{T}e^{ru}du-\int_{T_0}^{T}e^{rT_0}du\right) \\ &= S(t)e^{-rT}\left(\frac{1}{r}\left(e^{rT}-e^{rT_0}\right)-(T-T_0)e^{rT_0}\right) \\ &= \frac{S(t)}{r}\left[1-e^{-r(T-T_0)}\left(1+r(T-T_0)\right)\right].\end{aligned}$$

Remark. We have implicitly assumed $r \neq 0$. If $r = 0$, then $\Pi(t;X) = 0$ and

$$\lim_{r\to 0} = \frac{S(t)}{r}\left[1-e^{-r(T-T_0)}\left(1+r(T-T_0)\right)\right] = 0$$

as well.

Problem 3

(a) The LIBOR rate is given by

$$L(t,T) = \frac{1-p(t,T)}{(T-t)p(t,T)}.$$

To find the ZCB prices we use that the given model is an ATS model with

$$\alpha(t) = 0,\ \beta(t) = a,\ \gamma(t) = 0 \text{ and } \delta(t) = \sigma^2 t.$$

We get the ODE's

$$\begin{aligned}\frac{\partial B(t,T)}{\partial t} &= -1\\ B(T,T) &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial A(t,T)}{\partial t} &= aB(t,T) - \frac{\sigma^2 t}{2}B(t,T)^2\\ A(T,T) &= 0.\end{aligned}$$

From this

$$B(t,T) = T - t$$

and

$$\begin{aligned}\underbrace{A(T,T)}_{=0} &= A(t,T) + \int_t^T a(T-u)du - \frac{\sigma^2}{2}\int_t^T u(T-u)^2 du\\ &= A(t,T) + \frac{a}{2}(T-t)^2 - \frac{\sigma^2}{2}\left(T\frac{(T-t)^3}{3} - \frac{(T-t)^4}{4}\right).\end{aligned}$$

It follows that

$$A(t,T) = -\frac{a}{2}(T-t)^2 + \frac{\sigma^2}{2}\left(T\frac{(T-t)^3}{3} - \frac{(T-t)^4}{4}\right)$$

and finally

$$\begin{aligned}L(t,T) &= \frac{1-p(t,T)}{(T-t)p(t,T)}\\ &= \frac{1}{T-t}\left(\frac{1}{p(t,T)} - 1\right)\\ &= \frac{1}{T-t}\left(e^{-A(t,T)+B(t,T)r(t)} - 1\right)\\ &= \frac{1}{T-t}\left(e^{\frac{a}{2}(T-t)^2 - \frac{\sigma^2}{2}\left(T\frac{(T-t)^3}{3} - \frac{(T-t)^4}{4}\right) + (T-t)r(t)} - 1\right).\end{aligned}$$

(b) We use that

$$\Pi(t;X) = p(t,T)E^{Q^T}\left[X|\mathcal{F}_t\right],$$

where Q^T is the Q^T-forward measure. The price of the ZCB, $p(t,T)$, has been calculated in (a). If

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)v(t,T)dW^Q(t),$$

then the relation between W^Q and W^{Q^T} is given by

$$dW^Q(t) = v(t,T)dt + dW^{Q^T}(t).$$

In our case the short rate model has an ATS and it follows that

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)[-(T-t)\sigma\sqrt{t}]dW^Q(t).$$

Hence

$$v(t,T) = -(T-t)\sigma\sqrt{t},$$

and the dynamics of r under Q^T is given by

$$\begin{aligned} dr(t) &= adt + \sigma\sqrt{t}\left(-(T-t)\sigma\sqrt{t}\,dt + dW^{Q^T}(t)\right) \\ &= \left(a - \sigma^2 t(T-t)\right)dt + \sigma\sqrt{t}dW^{Q^T}(t). \end{aligned}$$

This can be written

$$\begin{aligned} r(T) &= r(t) + \int_t^T (a - \sigma^2 u(T-u))du + \int_t^T \sigma\sqrt{u}dW^{Q^T}(u) \\ &= r(t) + a(T-t) - \sigma^2\left(T\frac{(T-t)^2}{2} - \frac{(T-t)^3}{3}\right) + \int_t^T \sigma\sqrt{u}dW^{Q^T}(u). \end{aligned}$$

It follows that

$$E^{Q^T}\left[r(T)|\mathcal{F}_t\right] = r(t) + a(T-t) - \sigma^2\left(T\frac{(T-t)^2}{2} - \frac{(T-t)^3}{3}\right)$$

and

$$\text{Var}^{Q^T}(r(T)|\mathcal{F}_t) = \int_t^T \sigma^2 u\,du = \frac{\sigma^2}{2}(T^2 - t^2).$$

Since

$$E^{Q^T}\left[r(T)^2|\mathcal{F}_t\right] = \text{Var}^{Q^T}(r(T)|\mathcal{F}_t) + \left(E^{Q^T}\left[r(T)|\mathcal{F}_t\right]\right)^2$$

we get

$$\begin{aligned} \Pi(t;X) &= p(t,T)E^{Q^T}\left[r(T)^2|\mathcal{F}_t\right] \\ &= e^{A(t,T)-B(t,T)r(t)}\left(\frac{\sigma^2}{2}(T^2-t^2) + \left[r(t) + a(T-t) - \sigma^2\left(T\frac{(T-t)^2}{2} - \frac{(T-t)^3}{3}\right)\right]^2\right) \end{aligned}$$

with $A(t,T)$ and $B(t,T)$ as in (a).

Problem 4

The Q-dynamics of $(S_1(t), S_2(t))$ is given by

$$\begin{aligned} dS_1(t) &= rS_1(t)dt + S_1(t)\left[\sigma_{11}dW_1^Q(t) + \sigma_{12}dW_1^Q(t)\right] \\ dS_2(t) &= rS_2(t)dt + S_2(t)\left[\sigma_{21}dW_1^Q(t) + \sigma_{22}dW_2^Q(t)\right], \end{aligned}$$

where $W^Q = (W_1^Q, W_2^Q)$ is a two-dimensional Q-Wiener process.

(a) The arbitrage free price of X is given by

$$\begin{aligned} \Pi(t;X) &= e^{-r(T-t)}E^Q\left[\ln S_1(T) + \ln S_2(T)\big|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}E^Q\left[\ln S_1(T)|\mathcal{F}_t\right] + e^{-r(T-t)}E^Q\left[\ln S_2(T)|\mathcal{F}_t\right]. \end{aligned}$$

Now, under Q we can write

$$dS_1(t) \overset{D}{=} rS_1(t)dt + \sqrt{\sigma_{11}^2 + \sigma_{12}^2}S_1(t)d\hat{W}_1^Q(t)$$

and

$$dS_1(t) \overset{D}{=} rS_1(t)dt + \sqrt{\sigma_{21}^2 + \sigma_{22}^2}S_2(t)d\hat{W}_2^Q(t),$$

where $\hat{W}_1^Q$ and $\hat{W}_2^Q$ are two Q-Wiener processes. Since

$$S_1(T) = S_1(t)e^{(r-(\sigma_{11}^2+\sigma_{12}^2)/2)(T-t)+\sqrt{\sigma_{11}^2+\sigma_{12}^2}(\hat{W}_1^Q(T)-\hat{W}_1^Q(t))}$$

we get

$$\ln S_1(T) = \ln S_1(t) + (r - (\sigma_{11}^2 + \sigma_{12}^2)/2)(T-t) + \sqrt{\sigma_{11}^2 + \sigma_{12}^2}(\hat{W}_1^Q(T) - \hat{W}_1^Q(t))$$

from which it follows that

$$\begin{aligned} E^Q\left[\ln S_1(T)|\mathcal{F}_t\right] &= \ln S_1(t) + (r - (\sigma_{11}^2 + \sigma_{12}^2)/2)(T-t) + \sqrt{\sigma_{11}^2 + \sigma_{12}^2} E^Q\left[(\hat{W}_1^Q(T) - \hat{W}_1^Q(t))\middle|\mathcal{F}_t\right] \\ &= \ln S_1(t) + (r - (\sigma_{11}^2 + \sigma_{12}^2)/2)(T-t). \end{aligned}$$

A similar relation holds for $E^Q\left[\ln S_2(T)|\mathcal{F}_t\right]$, and combining both we get

$$\begin{aligned} \Pi(t;X) &= e^{-r(T-t)}\big[\ln S_1(t) + (r - (\sigma_{11}^2 + \sigma_{12}^2)/2)(T-t) \\ &\quad + \ln S_2(t) + (r - (\sigma_{21}^2 + \sigma_{22}^2)/2)(T-t)\big] \\ &= e^{-r(T-t)}\left[\ln S_1(t) + \ln S_2(t) + \left(2r - \frac{\sigma_{11}^2 + \sigma_{12}^2 + \sigma_{21}^2 + \sigma_{22}^2}{2}\right)(T-t)\right]. \end{aligned}$$

(b) Let

$$F(t, x_1, x_2) = \Pi(t;X) = e^{-r(T-t)}\left[\ln x_1 + \ln x_2 + \left(2r - \frac{\sigma_{11}^2 + \sigma_{12}^2 + \sigma_{21}^2 + \sigma_{22}^2}{2}\right)(T-t)\right].$$

The hedging portfolio is given by

$$\begin{aligned} h_i(t) &= \frac{\partial F}{\partial x_i}(t, S_1(t), S_2(t)),\ i = 1,2 \\ h^B(t) &= \frac{F(t, S_1(t), S_2(t) - \frac{\partial F}{\partial x_2}(t, S_1(t), S_2(t))S_1(t) - \frac{\partial F}{\partial x_2}(t, S_1(t), S_2(t))S_2(t)}{B(t)}. \end{aligned}$$

Now

$$\frac{\partial F}{\partial x_i}(t, S_1(t), S_2(t)) = e^{-r(T-t)} \cdot \frac{1}{x_i},\ i = 1,2$$

and we get

$$\begin{aligned} h_i(t) &= \frac{e^{-r(T-t)}}{S_i(t)},\ i = 1,2 \\ h^B(t) &= \frac{1}{e^{rt}} \cdot \left(e^{-r(T-t)}\left[\ln S_1(t) + \ln S_2(t) + \left(2r - \frac{\sigma_{11}^2 + \sigma_{12}^2 + \sigma_{21}^2 + \sigma_{22}^2}{2}\right)(T-t)\right]\right. \\ &\quad \left. - \frac{e^{-r(T-t)}}{S_1(t)} \cdot S_1(t) - \frac{e^{-r(T-t)}}{S_2(t)} \cdot S_2(t)\right) \\ &= e^{-rT}\left[\ln S_1(t) + \ln S_2(t) + \left(2r - \frac{\sigma_{11}^2 + \sigma_{12}^2 + \sigma_{21}^2 + \sigma_{22}^2}{2}\right)(T-t) - 2\right]. \end{aligned}$$

Problem 5

(a) We know that under Q we have

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW^Q(t),$$

were

$$dW^Q(t) = \frac{\alpha - r(t)}{\sigma}dt + dW(t)$$

is a Q-Wiener process. It follows that

$$\begin{aligned} dr(t) &= (b - ar(t))dt + c\left(-\frac{\alpha - r(t)}{\sigma}dt + dW^Q(t)\right) \\ &= \left(b - \frac{c\alpha}{\sigma} - \left(a - \frac{c}{\sigma}\right)r(t)\right)dt + cdW^Q(t). \end{aligned}$$

(b) We know that if

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)v(t,T)dW^Q(t),$$

then the relation between W^Q and W^{Q^T} is given by

$$dW^Q(t) = v(t,T)dt + dW^{Q^T}(t),$$

ehere W^{Q^T} is a Q^T-Wiener process. In our case the short rate model has an ATS, $p(t,T) = e^{A(t,T)-B(t,T)r(t)}$, and it follows that

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)[-cB(t,T)]dW^Q(t).$$

Hence

$$v(t,T) = -cB(t,T),$$

where $B(t,T)$ solves

$$\begin{aligned} \frac{\partial B(t,T)}{\partial t} - \left(a - \frac{c}{\sigma}\right) B(t,T) &= -1 \\ B(T,T) &= 0. \end{aligned}$$

The solution to this ODE is

$$B(t,T) = \frac{1}{a - c/\sigma}\left(1 - e^{-(a-c/\sigma)(T-t)}\right).$$

It follows that

$$dW^Q(t) = -\frac{c}{a - c/\sigma}\left(1 - e^{-(a-c/\sigma)(T-t)}\right) dt + dW^{Q^T}(t)$$

and we get

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma S(t)\left[-\frac{c}{a - c/\sigma}\left(1 - e^{-(a-c/\sigma)(T-t)}\right) dt + dW^{Q^T}(t)\right] \\ &= \left(r(t) - \frac{c\sigma}{a - c/\sigma}\left(1 - e^{-(a-c/\sigma)(T-t)}\right)\right) S(t)dt + \sigma S(t) dW^{Q^T}(t) \\ dr(t) &= \left(b - \frac{c\alpha}{\sigma} - \left(a - \frac{c}{\sigma}\right) r(t)\right) dt + c\left[-\frac{c}{a - c/\sigma}\left(1 - e^{-(a-c/\sigma)(T-t)}\right) dt + dW^{Q^T}(t)\right] \\ &= \left(b - \frac{c\alpha}{\sigma} - \left(a - \frac{c}{\sigma}\right) r(t) - \frac{c^2}{a - c/\sigma}\left(1 - e^{-(a-c/\sigma)(T-t)}\right)\right) dt + c dW^{Q^T}(t). \end{aligned}$$