EXAMINATION IN SF2975 FINANCIAL DERIVATIVES

Date: 2015-06-11, 14:00-19:00

Suggested solutions

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Problem 1

(a) The price of the deriviative $X = 1/\sqrt{S(T)}$ is given by

$$\Pi(t;X) = e^{-r(T-t)}E^Q\left[X|\mathcal{F}_t\right],$$

where S under Q has dynamics

$$dS(t) = (r-\delta)S(t)dt + \sigma S(t)dW^Q(t).$$

The solution to this SDE is

$$S(T) = S(t)e^{(r-\delta-\sigma^2/2)(T-t)+\sigma(W^Q(T)-W^Q(t))}.$$

Now

$$\frac{1}{\sqrt{S(T)}} = S(T)^{-1/2},$$

so

$$\begin{aligned}
\Pi(t;X) &= E^Q\left[e^{-r(T-t)}S(T)^{-1/2}|\mathcal{F}_t\right] \\
&= S(t)^{-1/2}e^{-r(T-t)}E^Q\left[e^{-\frac{1}{2}\left[(r-\delta-\sigma^2/2)(T-t)+\sigma(W^Q(T)-W^Q(t))\right]}|\mathcal{F}_t\right] \\
&= S(t)^{-1/2}e^{-r(T-t)-\frac{1}{2}(r-\delta-\sigma^2/2)(T-t)}\underbrace{E^Q\left[e^{-\frac{\sigma}{2}(W^Q(T)-W^Q(t))}\right]}_{=e^{\frac{\sigma^2}{8}(T-t)}} \\
&= S(t)^{-1/2}e^{\left(-\frac{3r}{2}+\frac{\delta}{2}+\frac{3\sigma^2}{8}\right)(T-t)}.
\end{aligned}$$

(b) The hedging portfolio $h(t) = (h^S(t), h^B(t))$, for $t \in [0,T]$, is given by

$$h^S(t) = \frac{\partial F}{\partial s}(t, S(t))$$

and

$$h^B(t) = \frac{F(t,S(t)) - S(t)\frac{\partial F}{\partial s}(t,S(t))}{B(t)},$$

where

$$F(t,s) = e^{-r(T-t)}E^Q_{t,s}\left[\frac{1}{\sqrt{S(T)}}\right] = s^{-1/2}e^{\left(-\frac{3r}{2}+\frac{\delta}{2}+\frac{3\sigma^2}{8}\right)(T-t)}.$$

It follows that

$$h^S(t) = -\frac{1}{2}S(t)^{-3/2}e^{\left(-\frac{3r}{2}+\frac{\delta}{2}+\frac{3\sigma^2}{8}\right)(T-t)}$$

and

$$h^B(t) = \frac{3}{2}S(t)^{-1/2}e^{\left(-\frac{3r}{2}+\frac{\delta}{2}+\frac{3\sigma^2}{8}\right)(T-t)-rt}.$$

Problem 2

The dynamics of $V^h(t)$ under P is given by

$$\begin{aligned} dV^h(t) &= \left(1-\sum_{i=1}^n u_i\right)\frac{V^h(t)}{B(t)}dB(t) + \sum_{i=1}^n u_i \frac{V^h(t)}{S_i(t)}dS_i(t) \\ &= \left(1-\sum_{i=1}^n u_i\right)V^h(t)rdt + \sum_{i=1}^n u_i V^h(t)(\alpha_i dt + \sigma_i dW_i(t)) \\ &= rV^h(t)dt + V^h(t)\sum_{i=1}^n u_i(\alpha_i - r)dt + V^h(t)\sum_{i=1}^n u_i\sigma_i dW_i(t). \end{aligned}$$

Since $V^h(t)$ is the value of a self-financing portfolio, under the EMM Q, where the bank account is numeraire, it will have dynamics

$$dV^h(t) = rV^h(t)dt + V^h(t)\sum_{i=1}^n u_i\sigma_i dW_i^Q(t),$$

where W^Q is an n-dimensional Wiener process. We can write

$$\sum_{i=1}^n u_i\sigma_i W_i^Q(t) = \sqrt{\sum_{i=1}^n u_i^2\sigma_i^2}\hat{W}(t) = \hat{\sigma}\hat{W}(t),$$

where $\hat{W}$ is a one-dimensional Q-Wiener process and

$$\hat{\sigma} := \sqrt{\sum_{i=1}^n u_i^2\sigma_i^2}.$$

Hence

$$dV^h(t) = rV^h(t)dt + \hat{\sigma}V^h(t)d\hat{W}(t).$$

We can solve this SDE to get

$$V^h(T) = V^h(t)e^{(r-\hat{\sigma}^2/2)(T-t)+\hat{\sigma}(\hat{W}(T)-\hat{W}(t))}.$$

Finally we get

$$
\begin{aligned}
\Pi(t;X) &= e^{-r(T-t)}E^Q\left[\left(V^h(T)\right)^\beta |\mathcal{F}_t\right] \\
&= e^{-r(T-t)}E^Q\left[(V^h(t))^\beta e^{\beta(r-\hat{\sigma}^2/2)(T-t)+\beta\hat{\sigma}(\hat{W}(T)-\hat{W}(t))}\right] \\
&= e^{-r(T-t)}(V^h(t))^\beta e^{\beta(r-\frac{\hat{\sigma}^2}{2})(T-t)+\frac{\beta^2\hat{\sigma}^2}{2}(T-t)} \\
&= (V^h(t))^\beta e^{\left[\beta(r-\frac{\hat{\sigma}^2}{2})+\frac{\beta^2\hat{\sigma}^2}{2}-r\right](T-t)},
\end{aligned}
$$

with $\hat{\sigma}$ as above.

Problem 3

(a) The dynamics of $f(t,T)$ under Q is given by

$$
df(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du\,dt + \sigma(t,T)dW^Q(t),
$$

where W^Q is a Q-Wiener process. Now

$$
\begin{aligned}
\int_t^T \sigma(t,u)du &= \sigma_0\int_t^T \left(1+a(u-t)e^{-b(u-t)}\right)du \\
&= \sigma_0\left(T-t+a\int_t^T (u-t)e^{-b(u-t)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_t^T (u-t)e^{-b(u-t)}du &= \{u-t=x \text{ and } du=dx\} \\
&= \int_0^{T-t} xe^{-bx}dx \\
&= \left[-\frac{x}{b}e^{-bx}\right]_0^{T-t} - \int_0^{T-t} -\frac{1}{b}e^{-bx}dx \\
&= -\frac{T-t}{b}e^{-b(T-t)} + \frac{1}{b}\left[-\frac{1}{b}e^{-bx}\right]_0^{T-t} \\
&= -\frac{T-t}{b}e^{-b(T-t)} + \frac{1}{b^2}\left(1-e^{-b(T-t)}\right) \\
&= \frac{1}{b^2}\left[1-e^{-b(T-t)}\left(1+b(T-t)\right)\right].
\end{aligned}
$$

Hence

$$
\int_t^T \sigma(t,u)du = \sigma_0\left(T-t+\frac{a}{b^2}\left[1-e^{-b(T-t)}\left(1+b(T-t)\right)\right]\right),
$$

and it follows that

$$\begin{aligned} df(t,T) &= \sigma_0^2\left(1+a(T-t)e^{-b(T-t)}\right)\cdot\left(T-t+\frac{a}{b^2}\left[1-e^{-b(T-t)}\left(1+b(T-t)\right)\right]\right)dt \\ &+\sigma_0\left(1+a(T-t)e^{-b(T-t)}\right)dW^Q(t). \end{aligned}$$

(b) The dynamics of $p(t,T)$ under Q is given by

$$dp(t,T)=r(t)p(t,T)dt-\left(\int_t^T\sigma(t,u)du\right)p(t,T)dW^Q(t),$$

where W^Q is a Q-Wiener process and $r(t)=f(t,t)$ is the short rate. We know from (a) that

$$\int_t^T\sigma(t,u)du=\sigma_0\left(T-t+\frac{a}{b^2}\left[1-e^{-b(T-t)}\left(1+b(T-t)\right)\right]\right),$$

so the dynamics is given by

$$dp(t,T)=r(t)p(t,T)dt-\sigma_0\left(T-t+\frac{a}{b^2}\left[1-e^{-b(T-t)}\left(1+b(T-t)\right)\right]\right)p(t,T)dW^Q(t).$$

Problem 4

See the book.

Problem 5

We know that in this kind of model, the dynamics of X is given by

$$dX(t)=(r_d-r_f)X(t)dt+\sigma_X X(t)dW^Q(t).$$

Here Q is the EMM where B_d is the numeraire. (To show this we use the fact that $B_f(t)X(t)/B_d(t)$ must be a Q-martingale – this give us the Girsanov kernel.) Under Q the exchange rate X is a geometric Brownian motion:

$$X(T)=X(t)e^{(r_d-r_f-\sigma_X^2/2)(T-t)+\sigma_X(W^Q(T)-W^Q(t))}.$$

Hence

$$\ln X(T)=\ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t)+\sigma_X(W^Q(T)-W^Q(t)),$$

and we have

$$\begin{aligned} E^Q\left[\ln X(T)|F_t\right] &= E^Q\left[\ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t)+\sigma_x(W^Q(T)-W^Q(t))|\mathcal{F}_t\right] \\ &= \ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t),\ \text{and} \\ E^Q\left[(\ln X(T))^2|\mathcal{F}_t\right] &= \mathrm{Var}^Q\left((\ln X(T))^2|\mathcal{F}_t\right)+\left(E^Q\left[\ln X(T)|\mathcal{F}_t\right]\right)^2 \\ &= \mathrm{Var}^Q\left(\ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t)+\sigma_X(W^Q(T)-W^Q(t))|\mathcal{F}_t\right)+ \\ &\quad \left(\ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t)\right)^2 \\ &= \sigma_X^2(T-t)+\left(\ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t)\right)^2. \end{aligned}$$

This yields

$$\begin{aligned}\Pi(t;Z) &= e^{-r_d(T-t)}E^Q\left[1+\ln X(T)+(\ln X(T))^2|\mathcal{F}_t\right]\\ &= e^{-r_d(T-t)}\left(1+E^Q\left[\ln X(T)|\mathcal{F}_t\right]+E^Q\left[(\ln X(T))^2|\mathcal{F}_t\right]\right)\\ &= e^{-r_d(T-t)}\Big(1+\ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t)\\ &\quad +\sigma_X^2(T-t)+\left(\ln X(t)+(r_d-r_f-\sigma_X^2/2)(T-t)\right)^2\Big).\end{aligned}$$