EXAMINATION IN SF2975 FINANCIAL DERIVATIVES

Date: 2016-10-21, 14:00-19:00

Suggested solutions

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Problem 1

(a) The price of the derivative $X = \mathbf{1}(\ln S(T) \geq 0)$ is given by

$$\Pi(t; X) = e^{-r(T-t)} E^Q \left[\mathbf{1}(\ln S(T) \geq 0) \,|\, \mathcal{F}_t\right] = e^{-r(T-t)} Q\left(\ln S(T) \geq 0 \,|\, \mathcal{F}_t\right).$$

Under Q

$$S(T) = S(T) e^{(r-\sigma^2/2)(T-t)+\sigma(W^Q(T)-W^Q(t))},$$

where W^Q is a Q-Wiener process. It follows that

$$\ln S(T) = \ln S(t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W^Q(T) - W^Q(t)) \geq 0$$

$$\Leftrightarrow$$

$$\frac{W^Q(T) - W^Q(t)}{\sqrt{T-t}} \geq \frac{-\ln S(t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Finally

$$\begin{aligned}
\Pi(t; X) &= e^{-r(T-t)} Q\left(\frac{W^Q(T) - W^Q(t)}{\sqrt{T-t}} \geq \frac{-\ln S(t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= e^{-r(T-t)} \left(1 - N\left(\frac{-\ln S(t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= e^{-r(T-t)} N\left(\frac{\ln S(t) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

where $N(x)$ is the distrbution function of a $N(0,1)$-distributed random variable.

(b) We know that with

$$F(t, S(t)) = \Pi(t; X) = e^{-r(T-t)} N\left(\frac{\ln S(t) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

we have

$$\begin{aligned}
h^S(t) &= \frac{\partial F}{\partial x}(t, S(t)) \\
h^B(t) &= \frac{F(t, S(t)) - S(t)\frac{\partial F}{\partial x}(t, S(t))}{B(t)}.
\end{aligned}$$

We get

$$h^S(t) = e^{-r(T-t)} \cdot \frac{1}{S(t)} \cdot \frac{1}{\sigma\sqrt{T-t}}\varphi\left(\frac{\ln S(t) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

and

$$\begin{aligned} h^B(t) &= e^{-rt}\left[e^{-r(T-t)}N\left(\frac{\ln S(t) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - e^{-r(T-t)}\varphi\left(\frac{\ln S(t) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \\ &= e^{-rT}\left[N\left(\frac{\ln S(t) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - \frac{1}{\sigma\sqrt{T-t}}\varphi\left(\frac{\ln S(t) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right]. \end{aligned}$$

where φ is the density function of a $N(0,1)$-distributed random variable.

Problem 2

(a) This is an ATS model with

$$\alpha(t) = 0,\ \beta(t) = a,\ \gamma(t) = 0 \text{ and } \delta(t) = \sigma_0^2(t+1).$$

The ZCB prices are given by

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)},$$

where $A(t,T)$ and $B(t,T)$ solves

$$\begin{aligned} \frac{\partial B}{\partial t}(t,T) &= -1 \\ B(T,T) &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial A}{\partial t}(t,T) &= aB(t,T) - \frac{1}{2}\sigma_0^2(t+1)B^2(t,T) \\ A(T,T) &= 0 \end{aligned}$$

respectively. We get

$$B(t,T) = T - t$$

and

$$\begin{aligned} \underbrace{A(T,T)}_{=0} - A(t,T) &= a\int_t^T (T-u)du - \frac{\sigma_0^2}{2}\int_t^T (u+1)(T-u)^2 du \\ &= a\frac{(T-t)^2}{2} - \frac{\sigma_0^2}{2}\left[(T+1)\frac{(T-t)^3}{3} - \frac{(T-t)^4}{4}\right] \end{aligned}$$

$\Rightarrow$

$$A(t,T) = \frac{\sigma_0^2}{2}\left[(T+1)\frac{(T-t)^3}{3} - \frac{(T-t)^4}{4}\right] - a\frac{(T-t)^2}{2}.$$

It follows that

$$p(t,T) = e^{\frac{\sigma_0^2}{2}\left[(T+1)\frac{(T-t)^3}{3} - \frac{(T-t)^4}{4}\right] - a\frac{(T-t)^2}{2} - r(t)(T-t)}.$$

(b) The dynamics of $p(t,T)$ under Q is given by

$$\begin{aligned}
dp(t,T) &= d\left(e^{A(t,T)-B(t,T)r(t)}\right) \\
&= (\cdots)dt - B(t,T)\sigma_0\sqrt{t+1}p(t,T)dW^Q(t) \\
&= \left\{B(t,T) = T-t \text{ and } p(t,T) \text{ is the price of a traded asset}\right\} \\
&= r(t)p(t,T)dt - (T-t)\sigma_0\sqrt{t+1}p(t,T)dW^Q(t).
\end{aligned}$$

We know that if

$$dp(t,T) = r(T)p(t,T)dt + v(t,T)p(t,T)dW^Q(t)$$

under Q, then the Radon-Nikodym derivative $L^T(t)$ satisfies

$$dL^T(t) = v(t,T)L^T(t)dW^Q(t)$$

and we get

$$dW^Q(t) = v(t,T)dt + dW^{Q^T}.$$

In our case $v(t,T) = -(T-t)\sigma_0\sqrt{t+1}$, so

$$dW^Q(t) = -(T-t)\sigma_0\sqrt{t+1}dt + dW^{Q^T}(t).$$

Hence, the dynamics of r under Q^T is given by

$$dr(t) = \left(a - \sigma_0^2(T-t)(t+1)\right)dt + \sigma_0\sqrt{t+1}dW^{Q^T}(t).$$

Problem 3

We can have money without getting any return on it, and this is like a bank account with $r = 0$. Under the EMM where this asset, i.e. having cash, is numeraire, the dynamics of τ is given by

$$d\tau(t) = [b - \lambda\sigma - a\tau(t)]dt + \sigma dW^Q(t),$$

where $\lambda \in \mathbb{R}$ is the constant market price of risk. The theoretical price of the T-claim X at time $t = 0$ is given by

$$\Pi(0;X) = E^Q\left[\tau(T)\right].$$

The dynamics of τ can be written

$$\tau(T) = \tau_0 + \int_0^T [b - \lambda\sigma - a\tau(u)]du + \sigma W^Q(T),$$

and taking Q-expectations gives

$$E^Q\left[\tau(T)\right] = \tau_0 + (b-\lambda\sigma)T - a\int_0^T E^Q\left[\tau(u)\right]du.$$

With

$$m(T) = E^Q[\tau(T)]$$

this can be written

$$m'(T) = (b - \lambda\sigma) - am(T) \text{ and } m(0) = \tau_0.$$

Now

$$\begin{aligned}
\frac{d}{dT}\left(e^{aT}m(T)\right) &= ae^{aT}m(T) + e^{aT}m'(T) \\
&= e^{aT}(am(T) + m'(T)) \\
&= e^{aT}(b - \lambda\sigma),
\end{aligned}$$

and we get

$$\begin{aligned}
E^Q[\tau(T)] &= m(T) \\
&= \tau_0 e^{-aT} + (b - \lambda\sigma)\int_0^T e^{-a(T-u)}du \\
&= \tau_0 e^{-aT} + \frac{b - \lambda\sigma}{a}\left(1 - e^{-aT}\right).
\end{aligned}$$

Equating the observed price with the theoretical yields

$$\Pi_0^\star = \Pi(0; X) = \tau_0 e^{-aT} + \frac{b - \lambda\sigma}{a}\left(1 - e^{-aT}\right),$$

and from this

$$\lambda = \frac{1}{\sigma}\left(b - a\frac{\Pi_0^\star - \tau_0 e^{-aT}}{1 - e^{-aT}}\right).$$

Problem 4

See the book and notes from the lectures.

Problem 5

We have

$$\Pi(0; X) = e^{-rT}E^Q\left[S^\beta(T)\right].$$

Under Q, the dynamics of S is

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW^Q(t),$$

where W^Q is a Q-Wiener process. The solution to this SDE is

$$\begin{aligned}
S(T) &= S(0)e^{\int_t^T\left(r - \frac{\sigma^2(u)}{2}\right)du + \int_0^T \sigma(u)dW^Q(u)} \\
&= S(0)e^{rT - \frac{1}{2}\int_0^T \sigma^2(u)du + \int_0^T \sigma(u)dW^Q(u)}.
\end{aligned}$$

and it follows that

$$S^\beta(T) = S^\beta(0)e^{r\beta T - \frac{1}{2}\int_0^T \beta\sigma^2(u)du + \int_0^T \beta\sigma(u)dW^Q(u)}.$$

We know that

$$\int_0^T \beta\sigma(u)dW^Q(u) \sim N\left(0, \sqrt{\int_0^T \beta^2\sigma^2(u)du}\right),$$

and using this we can write

$$\begin{aligned}
\Pi(0;X) &= e^{-rT}E^Q\left[S^\beta(T)\right] \\
&= e^{-rT}S^\beta(0)e^{r\beta T - \frac{1}{2}\int_0^T \beta\sigma^2(u)du}E^Q\left[e^{\int_0^T \beta\sigma(u)dW^Q(u)}\right] \\
&= \left\{\text{If } X \sim N(0,\sigma), \text{ then } E\left[e^X\right] = e^{\frac{\sigma^2}{2}}\right\} \\
&= S^\beta(0)e^{r(\beta-1)T}e^{\frac{\beta(\beta-1)}{2}\int_0^T \sigma^2(u)du}.
\end{aligned}$$

Finally using

$$\begin{aligned}
\int_0^T \sigma^2(u)du &= \int_0^T \sigma_0^2\left(1+e^{-\gamma u}\right)^2 du \\
&= \sigma_0^2\int_0^T \left(1+2e^{-\gamma u}+e^{-2\gamma u}\right)du \\
&= \sigma_0^2\left[T + \frac{2}{\gamma}\left(1-e^{-\gamma T}\right) + \frac{1}{2\gamma}\left(1-e^{-2\gamma T}\right)\right]
\end{aligned}$$

we get

$$\Pi(0;X) = S^\beta(0)e^{r(\beta-1)T + \frac{\beta(\beta-1)}{2}\sigma_0^2\left[T+\frac{2}{\gamma}\left(1-e^{-\gamma T}\right)+\frac{1}{2\gamma}\left(1-e^{-2\gamma T}\right)\right]}.$$