EXAMINATION IN SF2975 FINANCIAL DERIVATIVES

Date: 2015-03-17, 08:00-13:00

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Allowed technical aids: None.

Any notation must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. Write only on one side of the page.

Good luck!

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Problem 1

(a) Solve the PDE

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{b^2x^2}{2}\frac{\partial^2 F}{\partial x^2} + aF &= 0 \\ F(T,x) &= x \end{aligned}$$

on $[0,T]$ where $a, b \in \mathbb{R}$ are constants. (3 p)

(b) Let W be a Wiener process and fix $T > 0$. Determine the distribution of the random variable

$$X = \int_0^T u^2 dW(u).$$

(3 p)

(c) State and prove the put-call parity when there is a bank account with constant rate $r \geq 0$ and the stock pays the constant dividend yield $\delta > 0$.

(4 p)

Problem 2

Consider the standard Black-Scholes model with bank account dynamics

$$dB(t) = rB(t)dt \text{ with } B(0) = 1,$$

and stock price dynamics

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \text{ with } S(0) > 0,$$

where $r \geq 0$, $\alpha \in \mathbb{R}$ and $\sigma > 0$.

(a) Determine the arbitrage free price $\Pi(t;X)$ at time $t \in [0,T]$ of the T-claim

$$X = \left[\ln S(T)\right]^2.$$

(5 p)

(b) Determine the hedging portfolio of the claim in (a). (5 p)

Problem 3

(a) Consider the following model for the forward rates under the objective measure P:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(0,T) &= f^\star(0,T). \end{aligned}$$

Here W is a 1-dimensional Wiener process under P and the volatility function is given by

$$\sigma(t,T) = \frac{\sigma_0}{1 + a(T-t)},$$

where $\sigma_0, a > 0$. We assume that the market model is free of arbitrage. Determine the dynamics of the forward rates $f(t,T)$ and the dynamics of the bond prices $p(t,T)$ for $0 \leq t \leq T < \infty$ under the martingale measure Q. (7 p)

(b) Let $T_0, T_1, \ldots T_N$ be the resettlement times for an interest rate swap contract. Show that the swap rate at $t \in [0, T_0)$ is given by

$$\frac{p(t,T_0) - p(t,T_N)}{\sum_{k=1}^{N}(T_k - T_{k-1})p(t,T_k)}.$$

(3 p)

Problem 4

Consider the following model for the short rate:

$$dr(t) = \sigma_0 dW^Q(t).$$

Here $\sigma_0 > 0$ is a given constant volatility and W^Q a 1-dimensional Q-Wiener process.

(a) Calculate for every $0 \leq t \leq T < \infty$ the forward rates $f(t,T)$. (7 p)

(b) In order to be able to calculate the objective probability of $r(T)$ being negative, a constant market price of risk λ_0 is assumed. Calculate the probability $P(r(T) < 0)$ under this assumption. (3 p)

Problem 5

Consider a model with a domestic bank account having dynamics

$$dB_d(t) = r_d B_d(t)dt \text{ with } B(0) = 1,$$

and a foreign stock (denoted in the foreign currency) with price process

$$dS_f(t) = \alpha_f S_f(t)dt + \sigma_f S_f(t)dW(t) \text{ with } S_f(0) > 0.$$

The exchange rate is assumed to satisfy

$$dX(t) = \alpha_X X(t)dt + \sigma_X X(t)dW(t) \text{ with } X(0) > 0.$$

We assume that $r \geq 0$, $\alpha_f, \alpha_X \in \mathbb{R}$ and $\sigma_f, \sigma_X > 0$.

(a) Determine the dynamics of X under the equivalent martingale measure when the domestic bank account is the numeraire. (5 p)

(b) Determine for $t \in [0, T]$ the arbitrage free price $\Pi(t; Z)$ of the T-claim with payoff

$$Z = S_f(T)X(T).$$

(5 p)