Avd. Matematisk statistik

KTH Matematik

EXTRA PROBLEMS RELATED TO THE BLACK-LITTERMAN MODEL

Recall the Black-Litterman formula:

$$\mu = \pi + \Sigma P^T \left(\frac{\Omega}{\tau} + P\Sigma P^T \right)^{-1} (q - P\pi).$$

Uppgift 1

Consider a Markowitz model with a risk-free asset with return $r_f = 2\%$ and two risky assets with covariance matrix for the returns given by

$$\Sigma = \begin{bmatrix} 0.07 & 0.02 \\ 0.02 & 0.06 \end{bmatrix}.$$

Assume that the expected excess returns of your benchmark portfolio are

$$\pi = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix},$$

and the estimated volatility is 10%. The historically estimated risk premium of this portfolio is 5%.

a) Compute the adjusted expected excess returns of the portfolio you would hold if you would follow two advices with the views that, in the next year, Asset 1 will have an excess return of 7%, with 31% uncertainty, and Asset 2 will underperfom Asset 1 with 10%, with 45% uncertainty.

b) These views might be in strong contrast with the benchmark portfolio's returns π. A way to quantify this difference is to check whether the returns obtained in a) are outliers with respect to the benchmark portfolio's returns.

Check whether the returns obtained in a) are outliers at a 5% level with respect to (π, Σ).

Remark. You do not need to derive the expressions you use. Note that $\lambda_{0.025} = 1.96$, $\chi^2_{0.05}(2) = 5.99$ and $\chi^2_{0.95}(2) = 0.01$. You may take $\tau = 1$, if you use the Black-Litterman formula.

Uppgift 2

Consider a Markowitz model with a risk-less asset with return $r_f = 3\%$ and two risky assets with covariance matrix for the returns given by

$$\Sigma = \begin{bmatrix} 0.08 & 0.04 \\ 0.04 & 0.07 \end{bmatrix}.$$

Assume that the weights of your benchmark portfolio are

$$w_B = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix},$$

and the estimated volatility is 20%. The historically estimated risk premium of this portfolio is 7%. Use the Black-Litterman model to compute the portfolio you would hold if you have the view that, in the next year, Asset 2 will outperform Asset 1, and you believe that a 95% prediction interval for the difference between the return of Asset 2 and Asset 1 is given by $4\% \pm 1\%$. Make sur to motivate your choice of the parameters.

(Note that $\lambda_{0.025} = 1.9600$. You may choose $\tau = 0.1$).

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SOLUTION

Uppgift 1

a) The adjusted expected excess return vector is given by the Black-Litterman formula:

$$\mu = \pi + \Sigma P^T \left(\frac{\Omega}{\tau} + P\Sigma P^T\right)^{-1} (q - P\pi),$$

where, π is the vector of expected excess returns of the benchmark portfolio, τ is a constant P is the views matrix, Ω is the covariance matrix of these views and, finally, q is the vector of the expected returns of the portfolio based on the matrix P.
The model's parametrs to be determined in order to apply this formula are:

1) The covarinace matrix of the returns:

$$\Sigma = \begin{bmatrix} 0.07 & 0.02 \\ 0.02 & 0.06 \end{bmatrix}.$$

Its inverse matrix is

$$\Sigma^{-1} = \begin{bmatrix} 15.78 & -5.26 \\ -5.26 & 18.42 \end{bmatrix}$$

2) The benchmark portfolio's expected excess returns

$$\pi = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}.$$

3) For the views stated in the problem we have

$$P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \; q = \begin{bmatrix} 0.07 \\ -0.1 \end{bmatrix}$$

and

$$\Omega = \begin{bmatrix} (0.31)^2 & 0 \\ 0 & (0.45)^2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

Note that the matrix P is invertible with inverse matrix

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Applying the Black-Litterman formula, the adjusted expected excess return vector is

$$\mu = \begin{bmatrix} 0.16 \\ 0.36 \end{bmatrix}.$$

b) We should check whether the follwoing holds:

$$(\mu - \pi)^t \Sigma^{-1} (\mu - \pi) \geq \chi^2_{0.05}(2).$$

We have

$$(\mu - \pi)^t \Sigma^{-1} (\mu - \pi) = 0.0352 < 5.99 = \chi^2_{0.05}(2).$$

Thus, μ is not an outlier w.r.t. (π, Σ).

Uppgift 2

In order to apply the Black-Litterman formula as given in Problem 3, we have to determine the models parametrs.

1) The benchmark (or the equilibrium) portfolio with expected excess returns π is the optimal solution to

$$\max_w \left\{ w^T \pi - \frac{\delta}{2} w^T \Sigma w \right\},$$

where $w = (w_1, w_2)^T$ denotes the weights vector of the portfolio of risky assets. The solution, w_B satisfies

$$\pi = \delta \Sigma w_B.$$

Choosing δ =(risk premium)/(varinace of w_B), we get

$$\delta = 0.07/0.2^2 = 1.75.$$

Therefore, the equillibrium expected excess return vector is

$$\pi = 1.75 \begin{bmatrix} 0.08 & 0.04 \\ 0.04 & 0.07 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.0980 \\ 0.1015 \end{bmatrix}.$$

2) For the views stated in the problem we have

$$P = (-1, 1), \ q = 0.04, \ \Omega = (0.01)^2/\lambda^2_{0.025} = (0.01)^2/(1.96)^2 = 2.603 \cdot 10^{-5}.$$

Hence,

$$\Sigma P^T = \begin{bmatrix} -0.04 \\ 0.03 \end{bmatrix}, \quad P\Sigma P^T = 0.07, \quad q - P\pi = 0.0365,$$

and

$$P\Sigma P^T = 0.07, \ (\Omega/\tau + P\Sigma P^T)^{-1} = (2.603 \cdot 10^{-5}/0.01 + 0.07)^{-1} = (0.0726)^{-1} = 13.77.$$

Apply the Black-Litterman formula, the adjusted expected excess return vector is

$$\begin{aligned} \mu &= \pi + \Sigma P^T \left(\frac{\Omega}{\tau} + P\Sigma P^T \right)^{-1} (q - P\pi) \\ &= \begin{bmatrix} 0.0980 \\ 0.1015 \end{bmatrix} + \begin{bmatrix} -0.04 \\ 0.03 \end{bmatrix} \cdot 13.77 \cdot 0.0365 \\ &= \begin{bmatrix} 0.0779 \\ 0.1166 \end{bmatrix}. \end{aligned}$$

Finally, the weights of the portfolio you should hold are

$$w = (\delta\Sigma)^{-1}\mu = \frac{1}{1.75} \begin{bmatrix} 17.5 & -10 \\ -10 & 20 \end{bmatrix} \begin{bmatrix} 0.0779 \\ 0.1166 \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.88 \end{bmatrix}.$$