Homeworks – Portfolio Theory SF2976

Fall 2011

See the last page for more information.

1 Portfolio Resampling and Utility Maximization

1.1 Portfolio Resampling

This part will illustrate the use of resampling methods in Markowitz portfolio optimization. We will consider the problem of maximizing quadratic utility given the risk-aversion λ:

$$\begin{aligned} \max\ & w^{\mathrm{T}}\mu - \frac{\lambda}{2} w^{\mathrm{T}}\Sigma w \\ \text{s.t.}\ & w^{\mathrm{T}}\mathbf{1} = 1 \\ & w_i \geq 0. \end{aligned}$$

As usual, $w = (w_1, \ldots, w_n)^{\mathrm{T}}$ denotes a portfolio of n assets.

Estimate the mean and the covariance matrix of the series of log-returns retrieved from the course web page, $\hat{\mu}_0$ and $\hat{\Sigma}_0$.
Choose a value N and generate N new series of data by bootstrapping from the original series. Estimate the mean and the covariance matrix using these series, $\hat{\mu}_i$ and $\hat{\Sigma}_i$, $i = 1, \ldots, N$. Make histograms of estimates of some element of μ and Σ.

Write a program that solves the above optimization problem. For each i, solve the problem for 30 different values of λ, chosen so that the main part of the efficient frontier is covered. Do the same using the original estimates $\hat{\mu}_0$ and $\hat{\Sigma}_0$.
Make a plot of the portfolios in a $\sigma - \mu$ diagram, using the original estimates $\hat{\mu}_0$ and $\hat{\Sigma}_0$. The original efficient frontier should be visible. (The resampled portfolios are preferably represented by e.g. dots or stars.)
Calculate the "resampled efficient frontier" by averaging over the resampled portfolios for each λ. Compare with the original frontier in a $\sigma - \mu$ diagram.
Repeat the procedure for two additional values of N (one small and one large), to illustrate the dependence of the resampled frontiers on N (you will have to zoom in on some part of the frontier).
Comment on the above results.

1.2 Utility Maximization

This part deals with the approach of maximizing the expected utility obtained from investing in a portfolio. In general, the utility of an allocation will depend on the level of wealth of the investor, i.e. money not invested will affect the allocation. The utility will of course always depend on the invested amount, but we will normalize the portfolios just as in the Markowitz approach. To avoid the influence from initial wealth, we can work with the exponential utility function, which ranks allocations independently of the wealth level.
The expected utility of a portfolio $w = (w_1, \dots, w_n)^{\mathrm{T}}$ of the assets $r = (r_1, \dots, r_n)^{\mathrm{T}}$ is

$$EU(W_0 - \alpha + \alpha(1 + w^{\mathrm{T}} r)),$$

where α denotes the invested amount.
Show that the exponential utility function

$$U(x) = -e^{-cx}$$

ranks allocations independently of the initial wealth W_0.
We will assume that the vector of asset returns r is normally distributed with mean vector μ and covariance matrix Σ. We will also assume that the amount invested is 1. Show that

$$E(-e^{-cw^{\mathrm{T}} r}) = -e^{-c(w^{\mathrm{T}}\mu - \frac{1}{2} c w^{\mathrm{T}} \Sigma w)}$$

by using the explicit form of the probability density and completing the square in the exponent. Identify the certainty equivalent of the allocation w and motivate why the portfolio that maximizes the expected utility above is the one that maximizes the certainty equivalent. Use this to find an explicit expression for the normalized portfolio that maximizes the expected utility. Show that the resulting portfolio can be written on the form

$$w^* = \frac{\Sigma^{-1}}{c}(\mu - \mu_{MVP}\mathbf{1}) + w_{MVP}. \tag{1}$$

Comment on the relation between the above solution (1) and the solution to the optimization problem in Section 1.1.

2 Scenario Optimization

This part deals with alternatives to classical Markowitz portfolio optimization. We will study three different methods: Mean Absolute Deviation, Minimum Regret and Conditional Value-at-Risk. Their common characteristic is that they can be formulated as linear programs using historical data or *scenarios*. (We will only use historical data and not generate scenarios.)

For the rest of the homework, we let $r = (r_1, \dots, r_n)^{\mathrm{T}}$ denote a vector of log-returns and let $r^{(t)}$ denote an observation of this vector at time t. The sample mean is denoted by

$$\overline{r} = \frac{1}{T} \sum_{t=1}^{T} r^{(t)}.$$

2.1 Mean Absolute Deviation

As the name suggests, Mean Absolute Deviation (MAD) studies the expected absolute deviation of the portfolio return. This means that we use

$$E\left(|r_{\text{portfolio}} - E(r_{\text{portfolio}})|\right)$$

as our measure of risk. We estimate this quantity by replacing the expectations with arithmetic averages. To find an optimal portfolio w.r.t. this risk measure, we solve the problem

$$\begin{aligned} \min_{w} \ & \frac{1}{T} \sum_{t=1}^{T} |w^{\mathrm{T}}(r^{(t)} - \overline{r})| \\ \text{s.t. } & w^{\mathrm{T}} \overline{r} \geq r_{\text{target}} \\ & w^{\mathrm{T}} \mathbf{1} = 1 \\ & w_i \geq 0. \end{aligned}$$

This is equivalent to the linear program

$$\begin{aligned} \min_{w, d_1, \dots, d_T} \ & \frac{1}{T} \sum_{t=1}^{T} d_t \\ \text{s.t. } & d_t - w^{\mathrm{T}}(r^{(t)} - \overline{r}) \geq 0 \\ & d_t + w^{\mathrm{T}}(r^{(t)} - \overline{r}) \geq 0 \\ & w^{\mathrm{T}} \overline{r} \geq r_{\text{target}} \\ & w^{\mathrm{T}} \mathbf{1} = 1 \\ & w_i \geq 0, \end{aligned}$$

for $t = 1, \dots, T$ and $i = 1, \dots, n$.

2.2 Minimum Regret

Minimum Regret (MR) maximizes the minimum return for a set of return scenarios (historical data in our case). This can be accomplished by solving the following linear program.

$$
\begin{aligned}
\max_{w, R_{\min}} \quad & R_{\min} \\
\text{s.t.} \quad & w^{\mathrm{T}} r^{(t)} - R_{\min} \geq 0 \\
& w^{\mathrm{T}} \overline{r} \geq r_{\text{target}} \\
& w^{\mathrm{T}} \mathbf{1} = 1 \\
& w_i \geq 0,
\end{aligned}
$$

for $t = 1, \ldots, T$ and $i = 1, \ldots, n$.

2.3 Conditional VaR

Conditional VaR (Expected Shortfall, Conditional Tail Expectation, Tail VaR etc.) is the expected value of the $1 - \alpha\%$ worst outcomes of the portfolio. Optimizing a portfolio w.r.t. CVaR with scenarios can be done using linear programming. The CVaR(α)-optimal portfolios are found by solving

$$
\begin{aligned}
\min_{\mathrm{VaR}, w, d_1, \ldots, d_T} \quad & \mathrm{VaR} + \frac{1}{T} \frac{1}{1 - \alpha} \sum_{t=1}^{T} d_t \\
\text{s.t.} \quad & d_t + \mathrm{VaR} + w^{\mathrm{T}} r^{(t)} \geq 0 \\
& d_t \geq 0 \\
& w^{\mathrm{T}} \overline{r} \geq r_{\text{target}} \\
& w^{\mathrm{T}} \mathbf{1} = 1 \\
& w_i \geq 0,
\end{aligned}
$$

for $t = 1, \ldots, T$ and $i = 1, \ldots, n$.
Note that the program also calculates the α-VaR.

Write programs that solve the above problems, make plots of the different possible portfolios and comment. Also, report the min-MAD, min-MR and min-CVaR portfolios.

2.4 Real Data

Apply the methods above to some other data set of your choice. Write a short description of the data and where it can be downloaded. Make plots and compare the min-MAD, min-MR and min-CVaR portfolios.

Information about the homeworks

1. Satisfactory reports handed in on time with solutions to both homeworks will give 1.5 credits (LAB 1) as well as 10 points on Problem 1 on the first exam. A satisfactory solution to one part will give 5 points on Problem 1.a. on the first exam. A correction is then needed to obtain the credits (LAB 1). If none of the solutions are satisfactory, they must be corrected to obtain the credits (LAB 1) and no points will be awarded on the exam.

2. **The written reports must be handed in to Ali Hamdi before *15:00, November 15, 2011 (part 1)* and *15.00, November 25, 2011 (part 2).***

3. Data for the homeworks can be found on the course web page.

4. The homeworks may be done in groups of up to four students.

5. No late reports will be accepted.

6. No e-mailed reports will be accepted.

7. No questions regarding homeworks will be answered on the last day of hand-in.

8. The report must be typeset (e.g. using LaTeX or Word), structured and well-written.

9. The report should include the following:
   - an introduction with a short summary
   - results with illustrating figures and tables
   - clear and concise comments of the results and figures
   - an appendix with programming code.

10. Each group must also send an e-mail to Ali Hamdi containing programming code written in Matlab or R, that give results in agreement with the results in the report.

Good Luck!