

Barrier Options

A barrier option is an option whose payoff depends on whether the price of the underlying object reaches a certain barrier during a certain period of time. The purpose of the barrier option is to have a derivative that is similar to European calls and puts but at the same time cheaper.

The payoff of the barrier call/put is identical to the payoff of a European call/put if the barrier option is “active” at maturity. The construction will be explained by example. Let us begin with the

Down-and-out call option. Suppose the current spot price is S_0 , the strike is K , and the barrier is $L < S_0$. In this case the option is “active” at the issuing of the option. If the spot price stays above L for the whole time period, until maturity T , then the payoff is $\max(S_T - K, 0)$ (identical to a European call option). However, if at some time before T the spot price falls below L , then the option becomes “inactive” and will pay 0 at time T . In mathematical terms the payoff at time T is

$$\max(S_T - K, 0)I\{\min_{0 \leq t \leq T} S_t > L\}.$$

Down-and-in call option. Suppose the current spot price is S_0 , the strike is K , and the barrier is $L < S_0$. In this case the option is “inactive” at the issuing of the option. If the spot price stays above L for the whole time period, until maturity T , then the payoff is 0 because the option has remained inactive. However, if at some time before T the spot price falls below L , then the option becomes “active” and will pay $\max(S_T - K, 0)$ at time T . In mathematical terms the payoff at time T is

$$\max(S_T - K, 0)I\{\min_{0 \leq t \leq T} S_t \leq L\}.$$

Up-and-out put option. Suppose the current spot price is S_0 , the strike is K , and the barrier is $L > S_0$. In this case the option is “active” at the issuing of the option. If the spot price stays below L for the whole time period, until maturity T , then the payoff is $\max(K - S_T, 0)$ (identical to a European put option). However, if at some time before T the spot price rises above L , then the option becomes “inactive” and will pay 0 at time T . In mathematical terms the payoff at time T is

$$\max(K - S_T, 0)I\{\max_{0 \leq t \leq T} S_t < L\}.$$

Up-and-in put option. Suppose the current spot price is S_0 , the strike is K , and the barrier is $L > S_0$. In this case the option is “inactive” at the issuing of the option. If the spot price stays below L for the whole time period, until maturity T , then the payoff is 0 because the option has remained inactive. However, if at some time before T the spot price rises

above L , then the option becomes “active” and will pay $\max(K - S_T, 0)$ at time T . In mathematical terms the payoff at time T is

$$\max(S_T - K, 0)I\{\max_{0 \leq t \leq T} S_t \geq L\}.$$

Pricing Barrier Options. Recall that the no arbitrage assumption implies that the present price of a derivative with payoff X at time T can be written as $Z_T E[X]$. In the context of barrier options (and more generally, path dependent options) we will think of X as

$$X = f(S_0, S_1, \dots, S_T),$$

where $S_0, \dots, S_T$ is the path of the spot price. For a down-and-out call option the function f is given by

$$f(S_0, S_1, \dots, S_T) = \max(S_T - K, 0)I\{\min_{0 \leq t \leq T} S_t > L\},$$

and similar for the other options.

When considering a binomial model, all the paths through the tree can be enumerated. Since there are 2^T possible paths we will enumerate them by $\omega_1, \dots, \omega_{2^T}$ and write

$$S_0(\omega_j), S_1(\omega_j), \dots, S_T(\omega_j),$$

for the j th path. Each path is equally likely, because there is always probability $1/2$ to go up or down in the tree, the probability of each path is 2^{-T} . Therefore, the price of a down-and-out call option is

$$\begin{aligned} Z_T E \left[\max(S_T - K, 0) I\{\min_{0 \leq t \leq T} S_t > L\} \right] \\ = Z_T \sum_{j=1}^{2^T} \max(S_T(\omega_j) - K, 0) I\{\min_{0 \leq t \leq T} S_t(\omega_j) > L\} 2^{-T}. \end{aligned}$$

For the other barrier options you just need to use the corresponding payoff function.

Example

Consider Exercise 2(a) in Chapter IX. Suppose we want to price an up-and-out put option on the share price of the stock, with strike price $K = \$13.70$ and barrier $L = \$13.50$. Using the spot tree, we see that the barrier option has value 0 at nodes where the spot price is 13.5591, 13.8441, 14.1351, 13.5808, 14.4322, 13.8663. One can construct an “option tree”

where the value at each node represents the “value if active” at that node. Start by filling out the zeros. Then work your way backwards, just as you would do when pricing a standard European call option, and pay attention to the “zeros”. In this example the initial price of the up-and-out barrier option is \$0.30.