XIII: Exchange Options and Change-of-Numeraire

In this chapter we will revisit the "change-of-numeraire technique" from Chapter VII and use it to price exchange options of American type.

Let X be the value in a base currency (USD, say) of a product, which will be taken to be 1kg of coffee at time T. Let N be the value in the base currency of a means of payment, which will be taken to be one packet of cigarettes at time T. In Chapter VII we introduced the *numeriare distribution* P^N by

$$P^N(A) = \frac{P_0^{(T)}[I_A N]}{P_0^{(T)}[N]},$$

for all events A observed at T and we have the formula

$$P_0^{(T)}[X] = P_0^{(T)}[N] E^N\left[\frac{X}{N}\right].$$

Note that the expected value $E^N\left[\frac{X}{N}\right]$ can be interpreted as the forward price of 1 kg of coffee in the currency cigarettes. To see this, note that

$$E^N\left[\frac{X}{N}\right] = \frac{P_0^{(T)}[X]}{P_0^{(T)}[N]} = \frac{G_0^{(T)}[X]}{G_0^{(T)}[N]}.$$

where $G_0^{(T)}[X]$ and $G_0^{(T)}[N]$ are the forward prices in the base currency. A forward contract to buy 1 kg of coffee for $G_0^{(T)}[X]/G_0^{(T)}[N]$ packets of cigarettes at time T leads to to cash-flow $X - NG_0^{(T)}[X]/G_0^{(T)}[N]$ at time T and 0 at time 0. The same cash-flow can be obtained by entering into a long position in forward contract on X and a short position in $G_0^{(T)}[X]/G_0^{(T)}[N]$ number of forward contracts on N.

The martingale property

Let us introduce the forward price process $Y_0, Y_1, \ldots, Y_T$ where $Y_t = G_t^{(T)}[X]/G_t^{(T)}[N]$. The next result shows that $\{Y_t\}$ is a P^N-martingale.

Theorem 1

The forward prices $\{Y_t\}$ have the martingale property w.r.t. the numeraire distribution P^N. In particular $Y_t = E_t^N[X/N]$ for $t \leq T$.

Proof

Consider the following strategy: Let A be any event whose outcome is known at time t. At time t enter a long position to buy 1kg of coffee for Y_t packets of cigarettes if A has occurred, otherwise do nothing. At time T collect $I_A(X - Y_t N)$ (which may be negative). The present price is zero, hence

$$0 = P_0^{(T)}[I_A(X - Y_t N)] = P_0^{(T)}[N] E^N [I_A(\frac{X}{N} - Y_t)].$$

Hence,

$$E^N [I_A \frac{X}{N}] = E^N [I_A Y_t],$$

which shows that $Y_t = E_t^N[\frac{X}{N}]$. This completes the proof.

European options

We have seen, in Chapter VII, how to use Black's model to price European options of exchange type. That is, the price of an option to buy crude oil for kg of gold. An alternative to Black's model is to use a binomial model. Here we will demonstrate how to price a European exchange option in a binomial model. In the next section we consider American exchange options.

The procedure is demonstrated by example. Consider a European option to sell, in 18 months from now, 15 barrels of crude oil for 1 Troy Ounce of gold. The forward price of crude oil is \$102.61 per barrel and the forward price of gold is \$1593.09 per Troy Ounce. The interest rate is $r = 2\%$ per six months. Let us determine the price of the European option in a binomial model with time step six months. Note that the payoff of the option at maturity is $15N \max(1/15 - X/N, 0)$, where N is the price of gold and X is the price of crude oil.

To construct a binomial tree, note that, if X denotes the price of crude oil at time T and N the gold price at time T, then $\{Y_t^1\}$ is a P^N-martingale, where

$$Y_t^1 = \frac{G_t^{(T)}[X]}{G_t^{(T)}[N]} = E_t^N \Big[\frac{X}{N}\Big].$$

Then we can construct a binomial tree for $\{Y_t^1\}$ as usual. That is, Y_0^1 is given and then $Y_{t+1}^1 = Y_t^1 u_1$ or $Y_{t+1}^1 = Y_t^1 d_1$, each with probability 1/2, where

$$u_1 = 1 + \tanh(\sigma_1 \sqrt{\Delta t}), \quad d_1 = 1 - \tanh(\sigma_1 \sqrt{\Delta t}).$$

The Y^1-tree the looks like

$$\begin{array}{llll} Y_0^1 & Y_0^1\,u & Y_0^1\,u^2 & Y_0^1\,u^3 \\ & Y_0^1\,d & Y_0^1\,du & Y_0^1\,du^2 \\ & & Y_0^1\,d^2 & Y_0^1\,d^2u \\ & & & Y_0^1\,d^3 \end{array}$$

Here, the tree represents the values of Y^1 at times $t_0 = 0$, $t_1 = 6$ months, $t_2 = 12$ months and $t_3 = 18$ months. Numerically, with $\sigma_1 =$ the Y^1-tree is

$$\begin{array}{llll} Y_0^1 & Y_0^1\,u & Y_0^1\,u^2 & Y_0^1\,u^3 \\ & Y_0^1\,d & Y_0^1\,du & Y_0^1\,du^2 \\ & & Y_0^1\,d^2 & Y_0^1\,d^2u \\ & & & Y_0^1\,d^3 \end{array}$$

The price, at time 0 of the option is given by

$$P_0^{(T)}[15N \max(1/15 - X/N, 0)] = 15P_0^{(T)}[N]E^N[\max(1/15 - Y_{t_3}^1, 0)].$$

Since $P_0^{(T)}[N] = Z_T G_0^{(T)}[N]$ is known we only need to compute the expected value $E^N[\max(1/15 - Y_{t_3}^1, 0)]$, which can be done with the tree for the conditional expectation $E_t^N[\max(1/15 - Y_{t_3}^1, 0)]$. Start at t_3, then we put the values of $\max(1/15 - Y_{t_3}^1, 0)]$ in the tree. Then we move to t_2. Since

$$E_{t_2}^N[\max(1/15 - Y_{t_3}^1, 0)] = \frac{1}{2}\Big(\max(1/15 - Y_{t_2}^1 u, 0) + \max(1/15 - Y_{t_2}^1 d, 0)\Big),$$

we get the values of $E_{t_2}^N[\max(1/15 - Y_{t_3}^1, 0)]$ as the average of the values one can reach from the corresponding point in the tree. Proceeding backwards in the same way give the tree for $E_t^N[\max(1/15 - Y_{t_3}^1, 0)]$ as

$$\begin{array}{llll} Y_0^1 & Y_0^1\,u & Y_0^1\,u^2 & Y_0^1\,u^3 \\ & Y_0^1\,d & Y_0^1\,du & Y_0^1\,du^2 \\ & & Y_0^1\,d^2 & Y_0^1\,d^2u \\ & & & Y_0^1\,d^3 \end{array}$$

Putting everything together we see that the price of the European option to sell oil for gold is $15XxX = \$1$.

American Exchange Options

In this section we continue the previous example, but consider the option to sell 15 barrels of crude oil for one Troy Ounce of gold as an American option, based on the futures price of crude oil and gold. That is, the option can be exercised before the maturity in 18 months.

In this case it is not sufficient to only consider $\{Y_t^1\}$ but one also needs to consider the *numeraire process* $Y_t^0 = 1/G_t^{(T)}[N]$. Note that, by

Theorem 1, $\{Y_t^0\}$ is also a P^N-martingale and we will need a joint model for the pairs $\{(Y_t^0, Y_t^1)\}$ so that both components are martingales. The way to construct the model is as follows. As before $Y_{t+1}^1 = Y_t^1 u_1$ or $Y_{t+1}^1 = Y_t^1 d_1$. Similarly we will assume that $Y_{t+1}^0 = Y_t^0 u_0$ or $Y_{t+1}^0 = Y_t^1 d_0$. The probability of the four possible moves are as follows:

$$\begin{array}{ll} P^N(u_0, u_1) = \rho/2, & P^N(u_0, d_1) = (1-\rho)/2, \\ P^N(d_0, u_1) = (1-\rho)/2, & P^N(d_0, d_1) = \rho/2, \end{array}$$

where ρ is a number in $(0, 1)$ that determines the dependence between the up/down moves. Note that $\rho = 1/2$ corresponds to independence.

The "tree" for the pair (Y_t^0, Y_t^1) can, at time t be represented as a square of size $(t+1) \times (t+1)$ in the following way. Initially at time t_0 we have $(Y_{t_0}^0, Y_{t_0}^1) = (1/G_0^{(T)}[N], G_0^{(T)}[X]/G_0^{(T)}[N])$. At time t_1 there are four possibilities that can be represented as a 2×2 square

$$\begin{array}{ll} (Y_{t_0}^0 d_0, Y_{t_0}^1 u_1) & (Y_{t_0}^0 u_0, Y_{t_0}^1 u_1) \\ \\ (Y_{t_0}^0 d_0, Y_{t_0}^1 d_1) & (Y_{t_0}^0 u_0, Y_{t_0}^1 d_1) \end{array}$$

At t_2 the possible states can be represented as a 3×3 square

$$\begin{array}{lll} (Y_{t_0}^0 d_0^2, Y_{t_0}^1 u_1^2) & (Y_{t_0}^0 u_0 d_0, Y_{t_0}^1 u_1^2) & (Y_{t_0}^0 u_0^2, Y_{t_0}^1 u_1^2) \\ \\ (Y_{t_0}^0 d_0^2, Y_{t_0}^1 u_1 d_1) & (Y_{t_0}^0 u_0 d_0, Y_{t_0}^1 u_1 d_1) & (Y_{t_0}^0 u_0^2, Y_{t_0}^1 u_1 d_1) \\ \\ (Y_{t_0}^0 d_0^2, Y_{t_0}^1 d_1^2) & (Y_{t_0}^0 u_0 d_0, Y_{t_0}^1 d_1^2) & (Y_{t_0}^0 u_0^2, Y_{t_0}^1 d_1^2) \end{array}$$

At t_3 the possible states can be represented as a 4×4 square

$$\begin{array}{llll} (Y_{t_0}^0 d_0^3, Y_{t_0}^1 u_1^3) & (Y_{t_0}^0 u_0 d_0^2, Y_{t_0}^1 u_1^3) & (Y_{t_0}^0 u_0^2 d_0, Y_{t_0}^1 u_1^3) & (Y_{t_0}^0 u_0^3, Y_{t_0}^1 u_1^3) \\ \\ (Y_{t_0}^0 d_0^3, Y_{t_0}^1 u_1^2 d_1) & (Y_{t_0}^0 u_0 d_0^2, Y_{t_0}^1 u_1^2 d_1) & (Y_{t_0}^0 u_0^2 d_0, Y_{t_0}^1 u_1^2 d_1) & (Y_{t_0}^0 u_0^3, Y_{t_0}^1 u_1^2 d_1) \\ \\ (Y_{t_0}^0 d_0^3, Y_{t_0}^1 u_1 d_1^2) & (Y_{t_0}^0 u_0 d_0^2, Y_{t_0}^1 u_1 d_1^2) & (Y_{t_0}^0 u_0^2 d_0, Y_{t_0}^1 u_1 d_1^2) & (Y_{t_0}^0 u_0^3, Y_{t_0}^1 u_1 d_1^2) \\ \\ (Y_{t_0}^0 d_0^3, Y_{t_0}^1 d_1^3) & (Y_{t_0}^0 u_0 d_0^2, Y_{t_0}^1 d_1^3) & (Y_{t_0}^0 u_0^2 d_0, Y_{t_0}^1 d_1^3) & (Y_{t_0}^0 u_0^3, Y_{t_0}^1 d_1^3) \end{array}$$

The options tree can be constructed in a similar manner as in Chapter IX. Start at the end, $t = t$ in the example, by representing the payoff of the option if exercised at t_4. The payoff at t_3 is

$$\max(N - 15X) = \frac{15}{Y_{t_3}^0} \max(\frac{1}{15} - Y_{t_3}^1, 0),$$

so in the options tree at t_3 the value at row $i+1$ (from the bottom) column $j+1$ (from the left), is

$$V_{t_3}^{i+1,j+1} = \frac{15}{Y_0^0 u_0^j d_0^{3-j}} \max\Big(\frac{1}{15} - Y_0^1 u_1^i d_1^{3-i}, 0\Big).$$

The value of the option at time t_2 is the maximum of the value of exercising the option and the value of continuing to hold it until t_3. The value of exercising at t_2 is

$$\max(G_{t_2}^{(t_3)}[N] - 15G_{t_2}^{(t_3)}[X]) = \frac{15}{Y_{t_2}^0} \max(\frac{1}{15} - Y_{t_2}^1, 0),$$

whereas the value of holding the option until t_3 is $P_{t_2}^{(t_3)}[V_{t_3}]$, where V_{t_3} is the value of the option at t_3. Then

$$\begin{aligned} P_{t_2}^{(t_3)}[V_{t_3}] &= P_{t_2}^{(t_3)}[N] E_{t_2}^N [V_{t_3}/N] \\ &= Z_{t_2,t_3} G_{t_2}^{(t_3)}[N] E_{t_2}^N [V_{t_3}/N] \\ &= Z_{t_2,t_3} \frac{1}{Y_{t_2}^0} E_{t_2}^N [Y_{t_3}^0 V_{t_3}]. \end{aligned}$$

The expected value $E_{t_2}^N[Y_{t_3}^0 V_{t_3}]$ at node $(i+1, j+1)$ in the t_2-tree, where $Y_{t_2}^0 = Y_0^0 u_0^j d_0^{2-j}$ and $Y_{t_2}^1 = Y_0^1 u_1^i d_1^{2-i}$ is then computed as

$$\begin{aligned} &P^N(u_0, u_1) Y_0^0 u_0^{j+1} d_0^{2-j} V_{t_3}^{i+2,j+2} + P^N(u_0, d_1) Y_0^0 u_0^{j+1} d_0^{2-j} V_{t_3}^{i+1,j+2} \\ &+ P^N(d_0, u_1) Y_0^0 u_0^{j+1} d_0^{1-j} V_{t_3}^{i+2,j+1} + P^N(d_0, d_1) Y_0^0 u_0^{i+1} d_0^{1-i} V_{t_3}^{i+1,j+1}. \end{aligned}$$

The value of the option at time t_1 is the maximum of the value of exercising the option and the value of continuing to hold it until t_2. The value of exercising at t_1 is

$$\max(G_{t_1}^{(t_3)}[N] - 15G_{t_1}^{(t_3)}[X]) = \frac{15}{Y_{t_1}^0} \max(\frac{1}{15} - Y_{t_1}^1, 0),$$

whereas the value of holding the option until t_2 is $P_{t_1}^{(t_2)}[V_{t_2}]$, where V_{t_2} is the value of the option at t_2. To compute this value, note that

$$\begin{aligned} P_{t_1}^{(t_2)}[V_{t_2}] &= P_{t_1}^{(t_3)}\Big[\frac{1}{Z_{t_2,t_3}} V_{t_2}\Big] \\ &= \frac{1}{Z_{t_2,t_3}} P_{t_1}^{(t_3)}[N] E_{t_1}^N [V_{t_2}/N] \\ &= \frac{Z_{t_1,t_3}}{Z_{t_2,t_3}} G_{t_1}^{(t_3)}[N] E_{t_1}^N \Big[V_{t_2} E_{t_2}^N [1/N]\Big] \\ &= \frac{Z_{t_1,t_3}}{Z_{t_2,t_3}} \frac{1}{Y_{t_1}^0} E_{t_1}^N [Y_{t_2}^0 V_{t_2}]. \end{aligned}$$

The expected value $E_{t_1}^N[Y_{t_2}^0 V_{t_2}]$ is computed similar to the above.

Proceeding backwards in a similar way until t_0 gives the value of the American exchange option at t_0.