KTH Mathematics

Exercise 1 in SF2701 Financial Mathematics, basic course, spring 2014.

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1. (a) The payoff at expiry for a European call option is

$$X = \phi(S_2) = \max\{S_2 - K, 0\}$$

At expiry we must have

$$\Pi(2, X) = X = \max\{S_2 - K, 0\}$$

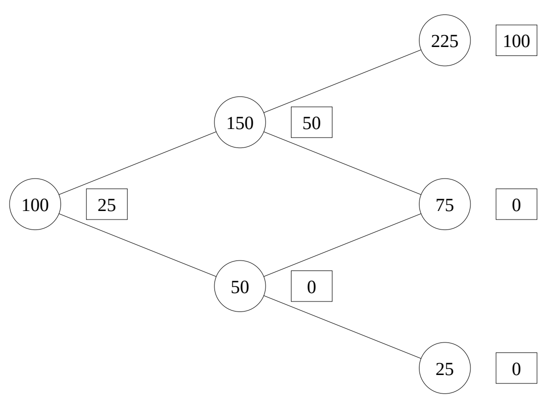
Now work backwards in the tree using risk neutral valuation. The martingale probabilities Q are found from

$$s = \frac{1}{1+r} E^Q[S_{t+1} | S_t = s].$$

and become

$$q = \frac{1 + r - d}{u - d} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes.



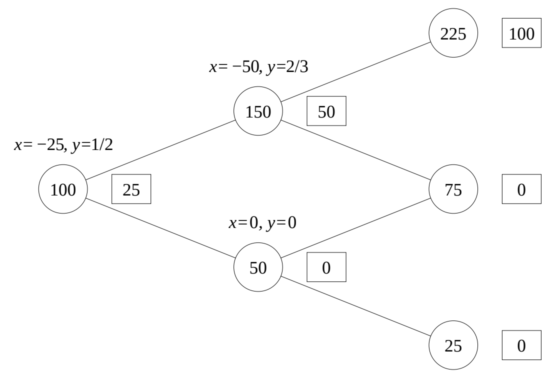
(b) To obtain the replicating portfolio at $t = 0$ we have to solve the following set of equations

$$\begin{cases} x + y \cdot 150 &= 50, \\ x + y \cdot 50 &= 0, \end{cases}$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This yields

$$x = -25, \qquad y = \frac{1}{2}.$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.



That the portfolio strategy is self-financing is seen from the following equations

$$\begin{cases} -25 + \frac{1}{2} \cdot 150 & = & -50 + \frac{2}{3} \cdot 150, \\ -25 + \frac{1}{2} \cdot 50 & = & 0 + 0 \cdot 50, \end{cases}$$

2. (a) The payoff at expiry for the binary asset-or-nothing is
At expiry we must have

$$\Pi(2, X) = X$$

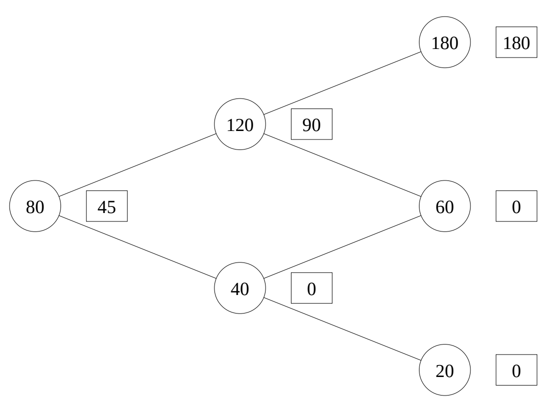
Now work backwards in the tree using risk neutral valuation. The martingale probabilities Q are found from

$$s = \frac{1}{1+r} E^Q[S_{t+1} | S_t = s].$$

and become

$$q = \frac{1 + r - d}{u - d} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes.



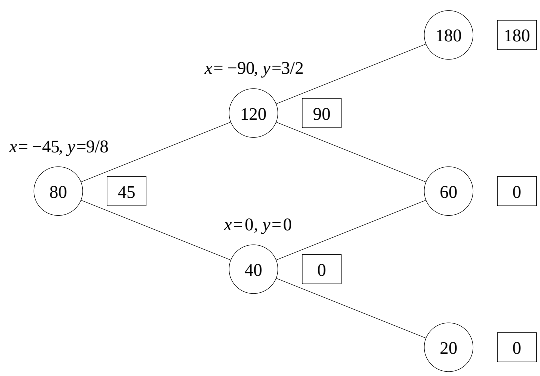
(b) To obtain the replicating portfolio at $t = 0$ we have to solve the following set of equations

$$\begin{cases} x + y \cdot 120 & = & 90, \\ x + y \cdot 40 & = & 0, \end{cases}$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This yields

$$x = -45, \qquad y = \frac{9}{8}.$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.



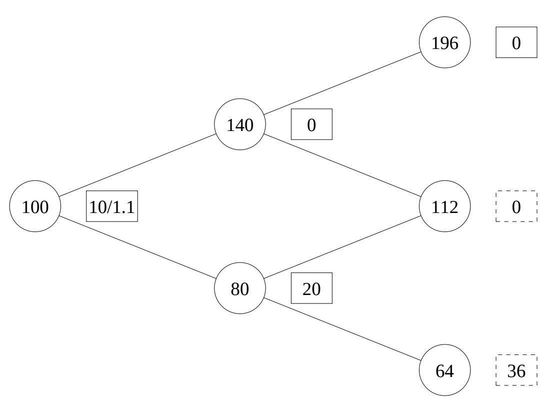
That the portfolio strategy is self-financing is seen from the following equations

$$\begin{cases} -45 + \frac{9}{8} \cdot 120 &= -90 + \frac{3}{2} \cdot 120, \\ -45 + \frac{9}{8} \cdot 40 &= 0 + 0 \cdot 40. \end{cases}$$

3. (a) For the martingale probabilities we have

$$q = \frac{1 + r - d}{u - d} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes. The value 20 adjacent to the node with stock price 80 is obtained as $\max\{100 - 80, \frac{1}{1.1}(0.5 \cdot 0 + 0.5 \cdot 36)\}$. Thus, an early exercise of the option is optimal at this node!



The price of the put option is thus 9.09 kr.

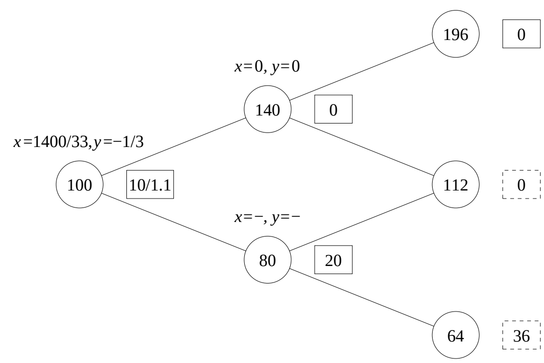
(b) To obtain the replicating portfolio at $t = 0$ we have to solve the following set of equations

$$\begin{cases} 1.1x + y \cdot 140 &= 0, \\ 1.1x + y \cdot 80 &= 20, \end{cases}$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This yields

$$x = \frac{1400}{33}, \qquad y = -\frac{1}{3}.$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.



Note that since the option is exercised at the node with stock price 80, you will from that node on no longer hold a portfolio.

That the portfolio strategy is self-financing is seen from the following equation

$$1.1 \cdot \frac{1400}{33} - \frac{1}{3} \cdot 140 = 0 + 0 \cdot 140.$$

4. 12.16 from Hull