1 A one period model

To have a concrete example in mind, suppose that we want to price a European call option on a stock that matures in six months.

1.1 The model setup

We will start simple with a one period model which means that we will only consider two points in time $t = 0$ (today) and $t = 1$ (six months from now).
Recall that arbitrage pricing is pricing in terms of what is already present on the market, so we need to specify what is already there.
We will assume that there is a *bond* with price process B (you can think of this as a bank account). The dynamics of the bond are given by

$$B(0) = 1, \qquad B(1) = 1 + R,$$

where R is the one period interest rate.

Also present on the market is a stock with price process S. For the stock we assume the following dynamics:

$$\begin{aligned} S(0) &= s \\ S(1) &= \begin{cases} su & \text{with probability } p \\ sd & \text{with probability } 1-p \end{cases} \end{aligned}$$

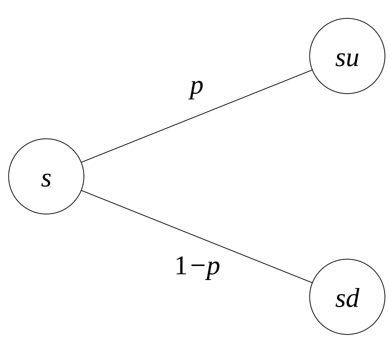
where we assume that $d < u$. This can also be written as

$$S(1) = sZ,$$

where Z is a random variable such that

$$P(Z = u) = p, \qquad P(Z = d) = 1 - p.$$

Stock:



1.2 Portfolios, arbitrage, derivatives and completeness

To find the correct price of the derivative we are going to compare the payoff of the derivative with the payoff of strategies already present in the market. These strategies are described by portfolios and their value processes.

Definition 1 *A* portfolio *h is a vector $h = (x, y)$, where*

$$\begin{aligned} x &= \text{number of SEK in the bank} \\ y &= \text{number of stocks you own} \end{aligned}$$

The value of the portfolio is given by the value process *V defined by*

$$V^h(t) = xB(t) + yS(t).$$

Remark 1 Note that $h \in \mathbb{R}^2$, which means that you are allowed to go short and can buy/sell as much as you want.

We are going to compare the derivative with the payoff of strategies already present on the market, and we do this to make sure that our pricing will be *consistent*. Mathematically this is formalized by requiring that there should exist no arbitrage opportunities on the market. The following definition states what we mean by arbitrage.

Definition 2 *A portfolio h is said to be an* arbitrage portfolio *if h is such that*

$$\begin{aligned} V^h(0) &= 0, \\ P(V^h(1) \geq 0) &= 1, \\ P(V^h(1) > 0) &> 0. \end{aligned}$$

A market is said to be free of arbitrage *if there exist no arbitrage portfolios.*

We have in mind to price a call option, but of course we want to be able to price all financial derivatives in the end (if possible). The definition of a financial derivative in this model is the following.

Definition 3 *A* financial derivative *or* contingent claim *is a random variable of the form*

$$X = \phi(S(1)), \text{ or equivalently, } X = \phi(Z).$$

The price at time t of a claim X *is denoted by $\Pi(t; X)$ or $\Pi_X(t)$. If there is no chance of misunderstanding we sometimes just write $\Pi(t)$.*

Example 1 To make our example more concrete suppose that we have the parameters $R = 0$, $u = 1.2$, $d = 0.8$, $p = 0.8$ and that the stock price today is $S(0) = 100$. This means that the stock price at time $t = 1$ will be

$$S(1) = \begin{cases} 120 & \text{with probability } 0.8 \\ 80 & \text{with probability } 0.2. \end{cases}$$

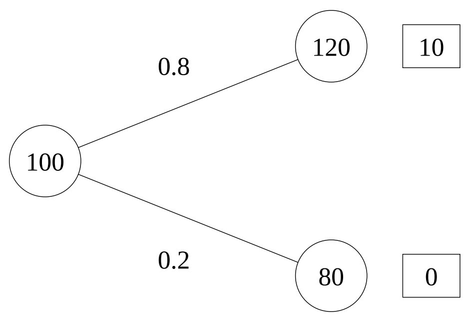
Also suppose that we want to price a European call option with strike price $K = 110$ (and exercise time $t = 1$). For this claim we have that

$$\begin{aligned} X &= \max\{S(1) - K, 0\} \\ &= \begin{cases} 120 - 110 = 10 & \text{if } S(1) = 120 \\ 0 & \text{if } S(1) = 80 \end{cases} \end{aligned}$$

Note that we have to have

$$\Pi(1; X) = X,$$

or there will be arbitrage! The problem is thus to find the price at time $t = 0$, $\Pi(0; X)$.



There are two common guesses for the price of the option at time $t = 0$:

Guess 1: The discounted expectation of future payoffs.

$$\Pi(1) = \frac{1}{1+0}\{0.8 \cdot 10 + 0.2 \cdot 0 = 8\}.$$

Guess 2: There is no correct price, because to determine the price we need to know more about the market.

The correct price is $\Pi(0; X) = 5$. □

How is the correct price obtained? Again, we should compare with already existing strategies!

Definition 4 *A claim X is said to be* reachable *if there is a portfolio h such that*

$$V^h(1) = X, \text{ with probability 1.}$$

The portfolio h is said to be a hedging *or* replicating portfolio *for the claim X.*
The market is said to be complete *if all claims are reachable.*

Example 2 Let us continue with Example 1. Consider the portfolio $h = \left(-20, \frac{1}{4}\right)$ where

$$\begin{aligned} x &= -20 \quad \text{borrow 20 SEK} \\ y &= \frac{1}{4} \quad \text{buy one fourth of a stock} \end{aligned}$$

Then we have that

$$\begin{aligned} V^h(1) &= x \cdot B(1) + y \cdot S(1) \\ &= \begin{cases} -20 \cdot 1 + \frac{1}{4} \cdot 120 = 10 & \text{if } S(1) = 120 \\ -20 \cdot 1 + \frac{1}{4} \cdot 80 = 0 & \text{if } S(1) = 80 \end{cases} \end{aligned}$$

thus h replicates the payoff of the option. □

The idea is of course that if X is replicated by h then we should have that $\Pi(t;X) = V^h(t)$ for $t = 0, 1$.

Proposition 1 *If the claim X is replicated by the portfolio h then all prices except for*

$$\Pi(t;X) = V^h(t),$$

will give rise to arbitrage opportunities.

Proof: Suppose that $\Pi(0;X) < V^h(0)$. Then at time $t = 0$ we buy the claim, sell the portfolio and put $V^h(0) - \Pi(0)$ in the bank, so the net position is zero.
At time $t = 1$ we will receive X, and we will have to pay $V^h(1) = X$ to the holder of the portfolio. This cancels, but we will still have $(1 + R)(V^h(0) - \Pi(0)) > 0$ in the bank. Thus we have created an arbitrage.
Check $\Pi(0;X) > V^h(0)$ yourself. $\square$

Example 3 Continuing on Example 2 we now find that the value at time $t = 0$ of the replicating portfolio $h = (-20, 1/4)$ is

$$\begin{aligned} V^h(0) &= x \cdot B(0) + y \cdot S(0) \\ &= -20 \cdot 1 + \tfrac{1}{4} \cdot 100 \quad = \quad 5. \end{aligned}$$

Thus a fair price for the option is 5. $\square$

It turns out that the simple model we are considering is complete.

Proposition 2 *The model is complete (all claims are reachable) if $u > d$ (actually $u \neq d$).*

Proof: Fix a claim $X = \phi(Z)$. Then we need to show that there exists a portfolio h such that

$$\begin{aligned} V^h(1) &= \phi(u) \text{ if } Z = u \\ V^h(1) &= \phi(d) \text{ if } Z = d. \end{aligned}$$

Writing out explicitly what this means we need to find $h = (x, y)$ such that

$$\begin{aligned} (1 + R)x + suy &= \phi(u) \\ (1 + R)x + sdy &= \phi(d) \end{aligned}$$

This is a linear system of equations, which has a unique solution if $u > d$. The solution is given by

$$\begin{aligned} x &= \frac{1}{1+R} \cdot \frac{u\phi(d) - d\phi(u)}{u - d}, \\ y &= \frac{1}{s} \cdot \frac{\phi(u) - \phi(d)}{u - d}. \end{aligned}$$

$\square$

1.3 Risk neutral valuation

In a complete model all claims X can be priced according to

$$\Pi(0; X) = V^h(0),$$

where the replicating portfolio $h = (x, y)$ is given by

$$\begin{aligned} x &= \frac{1}{1+R} \cdot \frac{u\phi(d) - d\phi(u)}{u-d}, \\ y &= \frac{1}{s} \cdot \frac{\phi(u) - \phi(d)}{u-d}. \end{aligned}$$

This means that

$$\begin{aligned} \Pi(0; X) &= xB(0) + yS(0) = x + sy \\ &= \frac{1}{1+R} \cdot \left\{ \frac{(1+R)-d}{u-d}\phi(u) + \frac{u-(1+R)}{u-d}\phi(d) \right\} \\ &= \frac{1}{1+R} \cdot \{q\phi(u) + (1-q)\phi(d)\}, \end{aligned}$$

where

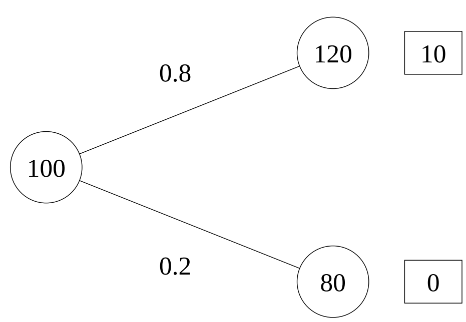
$$q = \frac{(1+R)-d}{u-d}.$$

Thus

$$\Pi(0; X) = \frac{1}{1+R} E^Q[X].$$

Guess 1 in Example 1 was not totally wrong after all. Prices can be computed as discounted expectations of future payoffs, but you should use the martingale probabilities Q, rather than objective probabilities P.

Example 4 Let us return to Example 1 one last time.



Using the stated parameters we find that

$$q = \frac{(1+R)-d}{u-d} = \frac{(1+0)-0.8}{1.2-0.8} = 0.5$$

Thus the correct price of the option is

$$\Pi(0; X) = \frac{1}{1+0}(0.5 \cdot 10 + 0.5 \cdot 0) = 5.$$

$\square$

1.4 Arbitrage revisited

We have been careful to check that we do not introduce arbitrage opportunities when pricing financial derivatives, but we have not checked that there were no arbitrage opportunities to begin with! Rest assured, under mild conditions the model is free of arbitrage.

Proposition 3 *The model is free of arbitrage if and only if* $d < (1+R) < u$.

The interpretation of the condition $d < (1+R) < u$ is the return on the stock can not dominate the return on the bond or vice versa.

Proof: We have to show that no arbitrage implies that $d < (1+R) < u$. We can instead show that if $d < (1+R) < u$ does not hold then there will be arbitrage.
Suppose therefore that $(1+R)s \geq u > d$, i.e. that the return on the bond dominates the return on the stock. Then the portfolio $h = (s, -1)$ is an arbitrage portfolio, since

$$V^h(0) = sB(0) + (-1)S(0) = s - s = 0,$$

and

$$V^h(1) = sB(1) + (-1)S(1) = s(1+R) - sZ \geq 0$$

with probability one, and with probability $1-p$ we have that

$$V^h(1) = s(1+R) - sd > 0.$$

Check $(1+R) \leq d < u$ yourself.

Now we have to show that if $d < (1+R) < u$, then there is no arbitrage. So suppose that $d < (1+R) < u$ and show that then there exist no arbitrage portfolios.
Fix an arbitrary portfolio $h = (x, y)$ such that $V^h(0) = 0$, i.e.

$$V^h(0) = xB(0) + yS(0) = 0.$$

This yields $x = -ys$. Now we have to show that the value process of h can not fulfill $V^h(1) \geq 0$ and $P(V^h(1) > 0) > 0$. We have

$$V^h(1) = xB(1) + yS(1) = ys(Z - (1+R)),$$

where we have used that $x = -ys$. If $y > 0$ then $V^h(1) \geq 0$ if

$$u - (1+R) \geq 0 \quad \text{and} \quad d - (1+R) \geq 0$$

which can not happen according to the assumption. Thus, if $y > 0$ then h is not an arbitrage portfolio. The case $y < 0$ is treated in the same way. □

The condition

$$d < 1 + R < u$$

implies that $1+R$ is a convex combination of d and u, i.e. that

$$1 + R = q \cdot u + (1-q) \cdot d \qquad \text{for some } 0 < q < 1.$$

We can interpret q as a probability. Introduce a new probability measure Q which is such that $Q(Z = u) = q$ and $Q(Z = d) = 1 - q$. Then

$$E^Q[S(1)] = q \cdot su + (1 - q) \cdot sd = s(1 + R),$$

so

$$S(0) = \frac{1}{1 + R} E^Q[S(1)].$$

The stock satisfies the risk neutral valuation formula! As a matter of fact this uniquely defines the martingale probabilities.

Definition 5 *A probability measure Q is said to be a* martingale measure *(*risk neutral measure*) if*

1. $0 < q < 1$ *(means $Q \sim P$)*
2. $$S(0) = \frac{1}{1 + R} E^Q[S(1)].$$

Remark 2 For our model we will get (as before)

$$q = \frac{(1 + R) - d}{u - d}.$$

We now summarize or findings concerning arbitrage pricing:

- We compute the price **as if** we live in a risk neutral world.

  Note: We use martingale probabilities!
- This does **not** mean that we live in a risk neutral world (or that we think that we do).
- The valuation formula is valid regardless of your attitude towards risk, as long as you prefer more risk less profit to less.

  The valuation formula is therefore often referred to as preference free.