1 Interest rates and bonds

1.1 Compounding

There are different ways of measuring interest rates

Example 1 The interest rate on a one-year deposit is 10% per annum. This statement means different things depending on the compounding frequency.

- With annual compounding it means that \$100 grows to

$$\$100 \times (1 + 0.10) = \$110$$

  by the end of the year.
- With semiannual compounding it means that \$100 grows to

$$\$100 \times \left(1 + \frac{0.10}{2}\right) \times \left(1 + \frac{0.10}{2}\right) = \$110.25$$

  by the end of the year.
- With compounding m times per year it means that \$100 grows to

$$\$100 \times \left(1 + \frac{0.10}{m}\right)^m$$

  by the end of the year.
- With continuous compounding (let $m \to \infty$) it means that \$100 grows to

$$\$100 \times e^{0.10 \cdot 1} \approx \$110.5171$$

  by the end of the year.
  For most practical purposes continuous compounding can be thought of as being equivalent to daily compounding, which here means that \$100 grows to

$$\$100 \times \left(1 + \frac{0.10}{365}\right)^{365} \approx \$110.5156$$

  by the end of the year.

□

Let r_c be a rate of interest with continuous compounding (quoted annually) and R_m the equivalent rate with compounding m times per year. Then

$$e^{r_c \cdot 1} = \left(1 + \frac{R_m}{m}\right)^m,$$

so

$$r_c = m \ln \left(1 + \frac{R_m}{m}\right)$$

and

$$R_m = m \left(e^{r_c/m} - 1\right).$$

1.2 Zero coupon bonds and zero rates

As stated before the most basic interest rate derivatives are zero coupon bonds.

Definition 1 *A* zero coupon bond *with \$1 principal and maturity T is a T-claim paying \$1 at time T.*
The price at time t of the bond is denoted $p(t, T)$ and we will assume that $p(T, T) = 1$.

Spot rates are defined from zero coupon bond prices. They are also known as zero coupon bond rates, or zero rates.

Definition 2 *The* (continuously compounded) spot rate $r(t, T)$ *at time t for the time interval $[t, T]$ is defined by*

$$p(t, T) = e^{-r(t,T)(T-t)}$$

or

$$r(t, T) = -\frac{\ln p(t, T)}{T - t}.$$

For each maturity T there exists a separate spot rate. The collection of all time t spot rates $\{r(t, T) : T = t, \ldots, \infty\}$ is referred to as the *term structure* (or zero coupon yield curve, or zero curve, or yield curve) *at time t.*

Example 2 Given prices of zero coupon bonds

T	$p(0, T)$
0.3	\$0.9851
0.6	\$0.9531
0.8	\$0.9231

the term structure is

T	$r(0, T)$
0.3	5%
0.6	8%
0.8	10%

□

1.3 Money market account

Before the money market account, or risk less asset, has been given by

$$B_T = e^{rT} \qquad (\text{if } B_0 = 1)$$

where r has been constant, but now we will have (in discrete time)

$$B_{t+\Delta t} = B_t e^{r(t,t+\Delta t)\cdot\Delta t}$$

so if we divide the time interval $[t, T]$ into n intervals of size $\Delta t = (T - t)/n$

$$B_T = B_t e^{\sum_{i=0}^{n-1} r(t+i\Delta t, t+(i+1)\Delta t)\cdot\Delta t}$$

Here $r(t, t + \Delta t)$ is the "over night" rate, and it is determined at time t.

Example 3 Consider a three month investment of \$10 in a money market account with monthly capitalization (means that Δt= 1 month = 1/12 year). Assume that the one month rate is 5% per annum the first month, 5.5% the second, and 6% the third. Then

$$\begin{aligned}
B_{3/12} &= B_0 e^{\sum_{i=0}^{3-1} r(0+i\Delta t, 0+(i+1)\Delta t)\cdot \Delta t} \\
&= 10 \exp\left\{ r(0, \frac{1}{12}) \cdot \frac{1}{12} + r(\frac{1}{12}, \frac{2}{12}) \cdot \frac{1}{12} + r(\frac{2}{12}, \frac{3}{12}) \cdot \frac{1}{12} \right\} \\
&= 10 e^{(0.05+0.055+0.06)\cdot \frac{1}{12}} = 10.1384.
\end{aligned}$$

Note that $r(1/12, 2/12)$ and $r(2/12, 3/12)$ are not really known at time $t = 0$! □

1.4 Fixed coupon bonds

Most bonds pay coupons to the holder periodically. The simplest coupon bond is the fixed coupon bond. The formal description is as follows.

- Fix times $T_0, T_1, \ldots, T_n$. Here T_0 is thought of as the emission date of the bond and $T_1, \ldots, T_n$ as coupon dates.
- At time T_i, $i = 1, \ldots, n$ the owner of the bond receives the deterministic coupon c_i.
- At time T_n the owner of the bond receives the principal (face value) K.

The coupon bond is simply a portfolio of zero coupon bonds. We have that the price of the bond for $t < T_1$ is

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + K p(t, T_n). \tag{1}$$

Often the coupons are determined in terms of return rather than monetary terms. The return for the i:th coupon is typically quoted as a simple rate acting on the principal K over $[T_{i-1}, T_i]$.

Here comes a small digression on simple rates. We have seen that the continuously compounded spot rate $r(t, T)$ for the interval $[t, T]$ solves

$$p(t, T) = e^{-r(t,T)(T-t)}$$

or

$$e^{r(t,T)(T-t)} = \frac{1}{p(t, T)}.$$

This is because an investment of \$1 at time t can create a payoff of $1/p(t, T)$ at time T. Just buy $1/p(t, T)$ T-bonds at time t, this will cost you exactly \$1, and at time T you will get the payoff $1/p(t, T)$.
The simple spot rate for $[t, T]$, henceforth referred to as LIBOR spot rate solves

$$1 + L(t, T)(T - t) = \frac{1}{p(t, T)},$$

and is defined as

$$L(t, T) = -\frac{p(t, T) - 1}{(T - t)p(t, T)}.$$

The zero rates and LIBOR spot rates are thus used to express the same return, they are just quoted in different ways.

Back to the coupon bonds. A simple rate r_i acting on K over $[T_{i-1}, T_i]$ results in

$$K[1 + r_i(T_i - T_{i-1})]$$

and by definition if the i:th coupon has a return r_i and the principal is K then

$$c_i = K r_i (T_i - T_{i-1}).$$

Remark 1 Recall that the rate R_m compounded m times per year makes \$1 grow to

$$\left(1 + \frac{R_m}{m}\right)^m$$

by the end of the year. Over a period of $1/m$ years \$1 grows to

$$\left(1 + R_m \cdot \frac{1}{m}\right)$$

which is the simple rate for the period $1/m$!

1.4.1 Determining zero rates by bootstrapping

Given market bond prices, how do we determine the zero coupon curve? Recall that most traded bonds are coupon bonds, particularly for the longer maturities. Since the price of a fixed coupon bond is made up of prices of zero coupon bonds, see Equation (1), which are in turn determined by the zero rates, there is a one-to-one correspondence, between bond prices and zero rates.

Starting from the short end of the term structure (that is short time to maturity) zero rates are obtained from zero coupon bonds. When moving to longer maturities bonds are coupon bonds, but now we can use that we have the short zero rates to back out the longer rates.

Example 4 Assume that in addition to the zero coupon bonds given in Example 2 there is also a coupon bond traded. This bond pays a \$5 annual coupon, has a principal of \$100, and matures in 1.6 years. The price of the coupon bond is \$92.82.

We can use this bond to find the 1.6-year zero rate. The price is given by

$$\begin{aligned} p_{fixed}(t) &= \sum_{i=1}^{n} c_i p(t, T_i) + K p(t, T_n) \\ &= 5e^{-r(t;t+0.6)0.6} + (5 + 100)e^{-r(t;t+1.6)1.6}. \end{aligned}$$

Now using that we know that $p_{fixed} = 92.82$ and that $r(t; t + 0.6) = 8\%$, we can solve for $r(t; t + 1.6)$ and obtain

$$r(t; t + 1.6) = -\frac{1}{1.6} \ln \frac{92.82 - 5e^{-0.08 \cdot 0.6}}{5 + 100} \approx 0.1100.$$

We can thus extend the table of zero rates to

$T - t$	$r(t, T)$
0.3	5%
0.6	8%
0.8	10%
1.6	11%

□

1.4.2 Yield and duration

Definition 3 *The* yield to maturity *of a fixed coupon bond is the interest rate y that when used to discount all coupons and the principal results in the market price. The yield thus solves*

$$p_{market}(t) = \sum_{i=1}^{n} c_i e^{-y(T_i - t)} + K e^{-y(T_n - t)}.$$

Remark 2 The yield to maturity of a zero coupon bond is simply the zero rate. This represents the bonds "internal rate of interest" and the above definition is an attempt to extend the concept to fixed coupon bonds.

Example 5 The yield to maturity of the coupon bond considered in Example 4 solves

$$92.82 = 5e^{-y \cdot 0.6} + (5 + 100)e^{-y \cdot 1.6}.$$

It should lie between 8% and 11% the zero rates for maturities 0.6 and 1.6, and it should be closer to 11%, since most weight is given to the large final payment.

Trial and error gives

$$y \approx 10.94\%.$$

□

To simplify notation include the principal K in the last coupon so $c_n^{new} = K + c_n^{old}$ and assume that $t = 0$. Also let $p = p_{fixed}$, then

$$p = \sum_{i=1}^{n} c_i e^{-y(T_i - t)}.$$

Definition 4 *The* duration D *of a fixed coupon bond is defined as*

$$D = \frac{\sum_{i=1}^{n} T_i c_i e^{-yT_i}}{p} = \sum_{i=1}^{n} T_i \frac{c_i e^{-yT_i}}{p}.$$

The duration is thus a weighted average of the coupon dates of the bond where the weight for a certain coupon date is the present value of the coupon payment divided by the value of the bond (which is the present value of all the payments). Duration is a measure of how long, on average, the holder of the bond has to wait before receiving cash payments or "mean time to coupon payment". Note that for a zero coupon bond with maturity T the duration is T. Duration also acts as a measure of sensitivity of the bond price to changes in the yield.

Proposition 1 *With notation as above we have*

$$\frac{dp}{dy} = \frac{d}{dy}\left\{\sum_{i=1}^{n} c_i e^{-y(T_i - t)}\right\} = -Dp.$$

So duration is essentially for bonds (with respect to yield) what delta is for derivatives (with respect to the underlying price). Approximately it holds that

$$\Delta p \approx -Dp\Delta y$$

or

$$\frac{\Delta p}{p} \approx -D\Delta y.$$

For concrete computations, see Example 6.5 in "Fundamentals of Futures and Options Markets", or Example 4.5 in "Options, Futures, and Other Derivatives".

Remark 3 Note that the change in yield is supposed to be the same for all maturities. This means that we are looking at what happens when there is a parallel shift in the yield curve (zero curve).

Corresponding to gamma for derivatives there is *convexity* for bonds. It is defined as

$$C = \frac{d^2 p}{dy^2}.$$

1.5 Forward rates

We have seen that the continuously compounded spot rate $r(t,T)$ solves

$$e^{r(t,T)(T-t)} = \frac{1}{p(t,T)}. \tag{2}$$

This is because an investment of \$1 at time t can create a risk less payoff of $\$1/p(t,T)$ at time T.
Now fix $t < S < T$ ans suppose t is today and we want to offer an interest rate, determined today, over the future interval $[S,T]$. We will extend the argument that led to (2).

- At time t sell an S-bond. This will earn you $p(t,S)$. Invest the money in T-bonds. This will give you $p(t,S)/p(t,T)$ T-bonds. The net investment at time t is zero.
- At time S you will have to pay \$1 to the owner of the S-bond.
- At time T you will receive $\$p(t,S)/p(t,T) \cdot 1$.

Thus an investment of \$1 at time S results in a payoff of $p(t,S)/p(t,T)$ at time T. We would therefore offer the rate

$$e^{f(t;S,T)(T-S)} = \frac{p(t,S)}{p(t,T)}. \tag{3}$$

Definition 5 *The* continuously compounded forward rate for $[S,T]$ contracted at t *is defined as*

$$f(t;S,T) = -\frac{\ln p(t,T) - \ln p(t,S)}{T-S} \tag{4}$$

Remark 4 Note that as $S \downarrow t$ the forward rate tends to the spot rate

$$\lim_{S \downarrow t} f(t;S,T) = r(t,T).$$

There exists a term structure of forward rates as well; it is in *two* dimensions since for each future time S there exists one forward rate for each $T \geq S$. Formally we have that the term structure of forward rates is given by $\{f(t;S,T) : \; S \in [t,\infty), \; T \in [S,\infty)\}$.
If we use that $p(t,T) = e^{-r(t,T)(T-t)}$ in (4) we get the following relationship between forward rates and spot rates

$$r(t,T)(T-t) = r(t,S)(S-t) + f(t;S,T)(T-S).$$

The term structure of forward rates is therefore determined by the term structure of spot rates. For a concrete example of computations using the relation, see Table 4.5 in Hull.