1 Pricing interest rate derivatives

1.1 General pricing formula

To do pricing we need the martingale measure Q. For interest rate models the definition is the following.

1. $Q \sim P$, and

2. the normalized price processes

$$Z(t,T) = \frac{p(t,T)}{B_t}, \qquad T > 0,$$

should be Q-martingales.

All interest rate models are incomplete, since the interest rate itself is not traded! There are thus infinitely many martingale measures and you always need market data to determine which one is being used on a particular market.
Once a martingale measure has been fixed all other normalized price processes (or if there are dividend payments, normalized gain processes) will also be Q-martingales, or there will be arbitrage. This results in the following pricing formula.

Proposition 1 *Let X be an interest rate derivative with maturity T. The price of X at time t is given by*

$$\begin{aligned}\Pi_t(X) &= B_t E^Q[\frac{X}{B_T}] \\ &= E^Q\left[e^{\sum_{n=t}^{T-1} r(n,n+1)} X \middle| \mathcal{F}_t \right]\end{aligned}$$

where we have used a time step of $\Delta t = 1$ for the bank account.

Remark 1 Note that we can not take the discounting factor out of the expectation since it is stochastic, and that usually the discount factor and X are dependent, so we can not compute the expectation as a product.

For zero coupon bonds we have that

$$p(t,T) = E^Q\left[e^{\sum_{n=t}^{T-1} r(n,n+1)} \cdot 1 \middle| \mathcal{F}_t \right] = e^{-r(t,T)(T-t)}.$$

1.2 Forward rate agreements (FRA)

A forward rate agreement is an agreement, at time $t = 0$, about a borrowing (lending) rate $\bar{r}_c$ (with continuous compounding), which will act on the principal K over the time interval $[S,T]$ in the future $0 < S < T$.
The cash flow of the borrower is

- At time S: $+K$
- At time T: $-Ke^{\bar{r}_c(T-S)}$

The cash flow of the lender is the negative of that of the borrower.
The value of a forward rate agreement at $t < S$ is (seen from the borrower's point of view)

$$\begin{aligned}\Pi_t &= p(t,S)K - p(t,T)Ke^{\bar{r}_c(T-S)} \\ &= K\left[p(t,S) - p(t,T)e^{\bar{r}_c(T-S)}\right].\end{aligned}$$

Usually $\bar{r}_c$ is set so as to make the value at time $t = 0$ zero. We then get

$$p(0,S) - p(0,T)e^{\bar{r}_c(T-S)} = 0$$

or

$$\begin{aligned}\bar{r}_c &= \frac{1}{T-S}\ln\frac{p(0,S)}{p(0,T)} \\ &= -\frac{\ln p(0,T) - \ln p(0,S)}{T-S} = f(0;S,T).\end{aligned}$$

So we should choose the forward rate contracted at time 0 for the interval $[S,T]$.
We know that the continuously compounded forward rate contracted at t for the interval $[S,T]$ solves

$$e^{f(t;S,T)(T-S)} = \frac{p(t,S)}{p(t,T)}.$$

Using this, the value of the forward rate agreement can be written

$$\begin{aligned}\Pi_t &= K\left(p(t,T)e^{f(t;S,T)(T-S)} - p(t,T)e^{\bar{r}_c(T-S)}\right) \\ &= p(t,T)K\left[e^{f(t;S,T)(T-S)} - e^{\bar{r}_c(T-S)}\right].\end{aligned}$$

If we instead use simple rates we first need the definition of forward LIBOR rates.

Definition 1 *The simple forward rate for $[S,T]$ contracted at t*

$$L(t;S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)}. \tag{1}$$

Remark 2 The LIBOR forward rate at time t for $[S,T]$ solves

$$[1 + L(t;S,T)(T-S)] = \frac{p(t,S)}{p(t,T)}.$$

The value of the forward rate agreement can now be written as

$$\begin{aligned}\Pi_t &= p(t,T)K\left\{1 + L(t;S,T)(T-S) - [1 + \bar{R}_s(T-S)]\right\} &(2)\\ &= p(t,T)K\left[L(t;S,T) - \bar{R}_s\right](T-S), &(3)\end{aligned}$$

where $\bar{R}_s$ is the constant rate over the time interval $[T-S]$ corresponding to $\bar{r}_c$, but now quoted as a simple rate.

1.3 Floating rate bonds

The coupons of a bond need not be fixed when the bond is issued. Usually they are determined by a financial benchmark, like a market interest rate, at some fixed date.
The simplest of these bonds is the one where the coupon rate r_i is set to the spot LIBOR rate $L(T_{i-1}, T_i)$. This means that

$$c_i = KL(T_{i-1}, T_i)(T_i - T_{i-1}).$$

We are now using coupon dates $T_1, \ldots, T_n$ and the principal K. The bond is issued at time T_0. Note that c_i is determined already at time T_{i-1}, but not paid out until time T_i. Assume that $T_i - T_{i-1} = \Delta$, $i = 1, \ldots, n$. Then if we use the definition of $L(T_{i-1}, T_i)$ from (1) (with $t = S = T_{i-1}$) we have that

$$\begin{aligned} c_i &= K \frac{1 - p(T_{i-1}, T_i)}{\Delta p(T_{i-1}, T_i)} \Delta \\ &= K \left(\frac{1}{p(T_{i-1}, T_i)} - 1 \right). \end{aligned}$$

The value of c_i at time $t < T_0$ is

$$\begin{aligned} \Pi_t(c_i) &= \Pi_t \left(K \left[\frac{1}{p(T_{i-1}, T_i)} - 1 \right] \right) \\ &= K \Pi_t \left(\frac{1}{p(T_{i-1}, T_i)} \right) - Kp(t, T_i). \end{aligned}$$

To find the value of $1/p(T_{i-1}, T_i)$ paid at time T_i try to replicate it.

- At time t buy a T_{i-1}-bond. This will cost $p(t, T_{i-1})$.
- At time T_{i-1} you will receive 1. Invest this in T_i-bonds. You will be able to buy $1/p(T_{i-1}, T_i)$ T_i-bonds.
- At time T_i you will receive $1/p(T_{i-1}, T_i)$.

Therefore the value at time t of obtaining $1/p(T_{i-1}, T_i)$ at time T_i is

$$\Pi_t \left(\frac{1}{p(T_{i-1}, T_i)} \right) = p(t, T_{i-1}).$$

The value of the i:th coupon at time t is then

$$\Pi_t(c_i) = K[p(t, T_{i-1}) - p(t, T_i)],$$

and the value of the floating rate bond at time t is thus

$$\begin{aligned} p_{float}(t) &= \sum_{i=1}^{n} K[p(t, T_{i-1}) - p(t, T_i)] + Kp(t, T_n) \\ &= Kp(t, T_0). \end{aligned}$$

In particular if $t = T_0$ we obtain $p_{float}(T_0) = K$. This means that the value of a floating rate bond immediately after a coupon payment is always K, and that the value of the bond in between coupon dates T_{i-1} and T_i is given by the value of obtaining $c_i + K$ at time T_i. Note that this is easy to compute since c_i is known already at time T_{i-1}. Therefore when $T_{i-1} \leq t < T_i$ we have

$$p_{float}(t) = (c_i + K)p(t, T_i).$$

1.4 Plain vanilla interest rate swaps

The simplest possible interest rate swap is the *fixed-for-floating* interest rate swap. It involves two parties: one, firm A, pays a floating rate (we will use the LIBOR spot rate) to firm B, that in turn pays a predetermined fixed rate, known as the *swap rate*, to firm A.
More precisely, we have a number of equally spaced dates $T_0, T_1, \ldots, T_n$ and payment occurs at the dates $T_1, \ldots, T_n$ (not at time T_0). Let $\Delta = T_i - T_{i-1}$, $i = 1, \ldots, n$ and denote the principal by K. Then at time T_i firm A will pay

$$KL(T_{i-1}, T_i)\Delta$$

and receive

$$K\bar{R}\Delta,$$

where $\bar{R}$ is the fixed swap rate. The net cash flow at time T_i is

$$K\left(L(T_{i-1}, T_i) - \bar{R}\right)\Delta.$$

Example 1 Consider a 2-year plain vanilla swap where the fixed rate $\bar{R} = 5.5\%$ with semi-annual compounding. The principal of the swap is \$1.000.000 and payments should be made semi-annually ($\Delta = 0.5$).

The prevailing LIBOR rate on January 1 is $L(0, 0.5) = 4.35\%$ with semi-annual compounding. This means that on July 1 firm A will pay firm B

$$\$1.000.000 \times 0.0435 \times 0.5 = \$21750,$$

and firm B will pay firm A

$$\$1.000.000 \times 0.055 \times 0.5 = \$27500.$$

In practice the net difference \$5750 will be paid (from firm B to firm A in this case). □

1.4.1 Valuing swaps

There are two (equivalent) approaches to value plain vanilla interest rate swaps, as bonds and as portfolios of forward rate agreements.

Valuing swaps as bonds With this view firm A has bought a fixed rate bond with return $\bar{R}$ issued by firm B. Thus firm A receives (fixed) coupon payments from firm B. Firm A has also issued a floating rate bond bought by firm B and has to pay coupons according to the prevailing spot rate. The value of the swap as seen from firm A's point of view is

$$\Pi_{swap}(t) = p_{fixed}(t) - p_{float}(t),$$

where p_{fixed} and p_{float} are the prices of the fixed and floating rate bonds, respectively.
Usually the swap rate $\bar{R}$ is chosen so that the value of the swap is initially approximately zero. This means that

$$\Pi_{swap}(T_0) = 0 = p_{fixed}(T_0) - p_{float}(T_0).$$

Now, we know that $p_{float}(T_0) = K$, so

$$Kp(T_0, T_n) + K\bar{R}\Delta\sum_{i=1}^{n} p(T_0, T_i) = K.$$

Solving for the swap rate $\bar{R}$ yields

$$\bar{R} = \frac{1 - p(T_0, T_n)}{\Delta \sum_{i=1}^{n} p(T_0, T_i)}.$$

Example 2 Consider the following 2-year term structure on January 1:

$T-t$	$r(t,T)$
0.5	4.3%
1.0	5.0%
1.5	5.1%
2.0	5.3%

Here rates are expressed with continuous compounding.

The value of the fixed coupon bond on January 1 is ($c_i = 27500,\ K = 1.000.000$)

$$\begin{aligned}
p_{fixed} &= \sum_{i=1}^{n} c_i p(t, T_i) + K p(t, T_n) \\
&= \sum_{i=1}^{n} c_i e^{-r(t,T_i)(T_i - t)} + K e^{-r(t,T_n)(T_n - t)} \\
&= 27500 e^{-0.043 \cdot 0.5} + 27500 e^{-0.05 \cdot 1.0} + 27500 e^{-0.051 \cdot 1.5} \\
&\quad + (27500 + 1.000.000) e^{-0.053 \cdot 2.0} \\
&\approx 1.002.707
\end{aligned}$$

The value of the floating rate bond is

$$\begin{aligned}
p_{float} &= (c_1 + K) p(t, T_1) \\
&= (21750 + 1.000.000) e^{-0.043034 \cdot 0.5} = 1.000.000 = K
\end{aligned}$$

which it should be right after emission. The value of the swap is therefore

$$\Pi_{swap}(t) = p_{fixed}(t) - p_{float}(t) = 2707$$

for firm A.

Now assume that the term structure on April 1 is the following:

$T-s$	$r(s,T)$
0.25	4.3%
0.75	5.1%
1.25	5.4%
1.75	5.7%

The value of the fixed coupon bond is now

$$\begin{aligned}
p_{fixed} &= \sum_{i=1}^{n} c_i p(s, T_i) + K p(s, T_n) \\
&= \sum_{i=1}^{n} c_i e^{-r(s,T_i)(T_i - s)} + K e^{-r(s,T_n)(T_n - s)} \\
&= 27500 e^{-0.043 \cdot 0.25} + 27500 e^{-0.051 \cdot 0.75} + 27500 e^{-0.054 \cdot 1.25} \\
&\quad + (27500 + 1.000.000) e^{-0.057 \cdot 1.75} \\
&\approx 1.009.332
\end{aligned}$$

The value of the floating rate bond is

$$\begin{aligned} p_{float} &= (c_1 + K)p(s, T_1) \\ &= (21750 + 1.000.000)e^{-0.043 \cdot 0.25} \approx 1.010.825 \end{aligned}$$

The value of the swap is now

$$\Pi_{swap}(s) = p_{fixed}(s) - p_{float}(s) = -1493$$

for firm A. □

Valuing swaps as portfolios of forward rate agreements We will show that each payment of the swap can be valued as a forward rate agreement. At time T_i the net cash flow for firm A is

$$K[L(T_{i-1}, T_i)\Delta - \bar{R}\Delta].$$

This can be written as

$$K[1 + L(T_{i-1}, T_i)\Delta - (1 + \bar{R}\Delta)].$$

Now if we use the definition of the LIBOR spot rate (Equation (1) with $t = S$), and transform $\bar{R}$ into its continuously compounded equivalent $\bar{r}_c$ we obtain

$$K\left\{\frac{1}{p(T_{i-1}, T_i)} - e^{\bar{r}_c \Delta}\right\}.$$

We have already seen that the value of $1/p(T_{i-1}, T_i)$ at time T_i is 1 at time T_{i-1} (replicate the payoff by an investment of 1 at time T_{i-1}). Therefore the net cash flow at time T_i of the swap can be seen as a certain cash flow of K at time T_{i-1} and a cash flow of $Ke^{\bar{r}_c \Delta}$ at time T_i, which is exactly the cash flow of a forward rate agreement.
This can also be seen from Equation (3)