1 Martingales

1.1 Basics

We begin directly with the definition of a martingale.

Definition 1 *A sequence $X = \{X_n\}_{n=1}^{\infty}$ of random variables is said to be a* martingale *with respect to the filtration $\underline{\mathcal{F}} = \{\mathcal{F}_n\}_{n\geq 0}$ (or an $\underline{\mathcal{F}}$-martingale) if for all $n \geq 0$*

1. *$X_n \in \mathcal{F}_n$ (i.e. X is adapted to $\underline{\mathcal{F}}$)*
2. *$E[|X_n|] < \infty$*
3. *$E[X_{n+1}|\mathcal{F}_n] = X_n$*

If we in 3 replace the equality with $\geq$ ($\leq$) X is said to be a submartingale *(*supermartingale*).*

Remark 1 As long as the sample space Ω is finite condition 2 is always fulfilled.

Example 1 Consider a sequence of tosses of a fair coin, and let

$$U_n = \begin{cases} 1 & \text{if the } n\text{:th toss is heads} \\ -1 & \text{if the } n\text{:th toss is tails} \end{cases}$$

You can think of U_n as the earnings if you bet \$1 on heads at coin toss number n. Now, let

$$X_n = \sum_{i=1}^{n} U_i, \qquad n \geq 1$$

then X_n represents your total earnings after n games betting on heads. Let $X_0 = 0$ and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$, $n \geq 1$.

Claim: X_n, $n \geq 0$ is a martingale with respect to $\underline{\mathcal{F}} = \{\mathcal{F}_n\}_{n\geq 0}$.

Check:

1. $X_0 = 0 \in \{\emptyset, \Omega\}$ and $X_n = \sum_{i=1}^{n} U_i \in \mathcal{F}_n = \sigma(U_1, \ldots, U_n)$ since it is a sum of $U_1, \ldots, U_n$, which are measurable.
2. $E[|X_n|] < \infty$ ($|X_n| \leq n$)
3. For the martingale property we have that

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n + U_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[U_{n+1}|\mathcal{F}_n] =$$

$$= X_n + E[U_{n+1}] = X_n + 0 = X_n.$$

where we have used the linearity of the conditional expectation to obtain the second equality, and that $X_n \in \mathcal{F}_n$, whereas U_{n+1} is independent of $\mathcal{F}_n$ to obtain the third.

Note that if the coin tossed is not fair so $P(U_n = 1) \leq 1/2$, then the above computations above give

$$E[X_{n+1}|\mathcal{F}_n] \leq X_n,$$

i.e. X_n, $n \geq 0$ is a supermartingale. In this case X_n corresponds to betting on an unfavorable game, so there is nothing "super" about a supermartingale. $\square$

Lemma 1 *If X_n, $n \geq 0$ is a martingale with respect to $\underline{\mathcal{F}}$ then*

$$E[X_n|\mathcal{F}_m] = X_m \qquad \text{for } n > m. \tag{1}$$

Proof: Suppose $n = m + k$, $k \geq 2$ ($k = 1$ is the martingale property). Then

$$E[X_{m+k}|\mathcal{F}_m] = E[E[X_{m+k}|\mathcal{F}_{m+k-1}]\mathcal{F}_m] = E[X_{m+k-1}|\mathcal{F}_m]$$

where we have used iterated expectations to obtain the first equality and property 3 of Definition 1. Now iterate the procedure and the result will follow. □

Remark 2 We could use (1) as the definition, but since it is more difficult to check than property 3 of Definition 1 we will not.

Lemma 2 *Property 3 of Definition 1 is satisfied if and only if*

$$E[\Delta X_n|\mathcal{F}_{n-1}] = 0 \qquad \text{for all } n \geq 1.$$

where $\Delta X_n = X_n - X_{n-1}$.

Proof: Writing things out we get

$$E[\Delta X_n|\mathcal{F}_{n-1}] = E[X_n - X_{n-1}|\mathcal{F}_{n-1}].$$

Now, using that the conditional expectation is linear, and that $X_{n-1} \in \mathcal{F}_{n-1}$, we obtain

$$E[\Delta X_n|\mathcal{F}_{n-1}] = E[X_n|\mathcal{F}_{n-1}] - X_{n-1} = 0.$$

The equivalence of the two properties should now be obvious. □

1.2 Martingale transforms

The goal of this section is to show that a discrete time stochastic integral preserves the martingale property. We start by looking at an example. The point of the example is to show that there is no system for beating a fair game (represented by a martingale).

Example 2 Let X_n, $n \geq 0$, be the martingale defined in Example 1. Recall that X_n was the amount of money you would have won betting \$1 each time on a fair game.

Now let H be a predictable process, i.e.

$$H_n \in \mathcal{F}_{n-1}.$$

H will represent our gambling strategy and thus for the n:th bet we may look at the outcomes at times $1, \ldots, n-1$, but not at time n, hence we require H_n to be predictable. Specifically, H_n should be the amount in \$ you bet at time n on heads.

Our winnings at time n can be expressed using a stochastic integral

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m \underbrace{(X_m - X_{m-1})}_{=U_m}$$

with the convention

$$(H \cdot X)_0 = 0.$$

□

Proposition 1 *Let X_n, $n \geq 0$ be a martingale, and H_n, $n \geq 1$ a predictable process such that $|H_n| \leq M$, $n \geq 1$. Then $(H \cdot X)_n$, $n \geq 0$ is a martingale.*

Proof: We need to check the conditions in Definition 1. We have the following.

1. $(H \cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1}) \in \mathcal{F}_n$, since it is a sum of products of measurable functions. Obviously $(H \cdot X)_0 = 0 \in \mathcal{F}_0$.

2. $$E[|(H \cdot X)_n|] \leq E\left[\sum_{m=1}^{n} |H_m||(X_m - X_{m-1})|\right] = \sum_{m=1}^{n} E[|H_m||(X_m - X_{m-1})|] \leq$$
$$\leq \sum_{m=1}^{n} ME[|X_m| + |X_{m-1}|] < \infty,$$
since X is a martingale.

3. We will check that $E[\Delta(H \cdot X)_n|\mathcal{F}_{n-1}] = 0$.
$$\begin{aligned} E[\Delta(H \cdot X)_n|\mathcal{F}_{n-1}] &= E[(H \cdot X)_n - (H \cdot X)_{n-1}|\mathcal{F}_{n-1}] \\ &= E[H_n(X_n - X_{n-1})|\mathcal{F}_{n-1}] \\ &= E[H_n \Delta X_n|\mathcal{F}_{n-1}] \\ &= H_n E[\Delta X_n|\mathcal{F}_{n-1}] = 0. \end{aligned}$$
Here we have used that $H_n \in \mathcal{F}_{n-1}$, since it is predictable, to obtain the second to last equality, and that X_n, $n \geq 0$ is a martingale to get the last equality.

□

Example 3 Going back to Example 2, in what way can we use Proposition 1 to deduce that you can not make money off a fair game? Proposition 1 tells us that $(H \cdot X)_n$, $n \geq 0$ is a martingale which means that

$$E[(H \cdot X)_n] = E[(H \cdot X)_0] = 0.$$

In words this says that our expected winnings at any time n are 0! □

Remark 3 The condition $|H_n| \leq M$, $n \geq 1$ is important, because otherwise the following strategy provides a "sure thing" when $P(U_i = 1) > 0$:

$$H_1 = 1, \quad \text{and } H_n = \begin{cases} 2H_{n-1} & \text{if } U_{n-1} = -1, \\ 1 & \text{if } U_{n-1} = 1, \end{cases}$$

which means that you should double every time you loose. If you loose k times and then win, your winnings will be

$$-1 - 2 - \ldots - 2^{k-1} + 2^k = 1.$$

Here obviously H_n is not bounded, and the mean loss just before the first head is

$$\frac{1}{2} \cdot 0 + \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{2^n}(1 + 2 + \ldots + 2^{n-1}) = \infty.$$

It would therefore be more accurate to say that there is no system for beating a fair game with limited resources.