1 A more general one period model

The reason for looking at this more general one period model is that it will allow us to state and prove the two fundamental theorems of mathematical finance without getting into to much technical detail. The theorems are valid for much more general models.

1.1 The model setup

We will once again consider a one period model which means that we will only consider two points in time $t = 0$ and $t = 1$. This model will live on a finite sample space $\Omega = \{\omega_1, \dots, \omega_M\}$ with a probability measure P defined on $\mathcal{F} = \mathcal{F}_1 = 2^\Omega$, such that $P(\omega_i) = p_i > 0$, $i = 1, \dots, M$.
There are $N + 1$ assets on the market and we will denote the price of asset number i at time t by S_t^i. Let

$$S_t = \begin{pmatrix} S_t^0 \\ S_t^1 \\ \vdots \\ S_t^N \end{pmatrix}$$

We will assume that S_0 is deterministic and that $S_1 \in \mathcal{F}$. Furthermore we will assume that asset number 0 is strictly positive, i.e. $S_0^0(\omega) > 0$, and $S_1^0(\omega) > 0$ for every $\omega \in \Omega$. This will allow us to use it as a *numeraire* asset.

1.2 Absence of arbitrage

Using the asset 0 as a *numeraire* will allow us to compare money today, $t = 0$, to money in the future, $t = 1$.

Definition 1 *The* normalized price process *is defined as*

$$Z_t = \frac{S_t}{S_t^0} = \begin{pmatrix} S_t^0/S_t^0 \\ S_t^1/S_t^0 \\ \vdots \\ S_t^N/S_t^0 \end{pmatrix}.$$

Remark 1 If S^0 is a bank account we are simply discounting everything to present value. Note that in the normalized economy we have that $Z^0 \equiv 1$, which means that it corresponds to a bank with zero interest rate.

As usual we need the concepts of a portfolio, the corresponding value process, and an arbitrage strategy.

Definition 2 *A* portfolio *is a vector* $h = (h^0, h^1, \dots, h^N)^T$*, where the super-index* T *denotes transpose.*
The value process V^S *corresponding to a portfolio* h *is defined as*

$$V^S(t) = \sum_{i=0}^{N} h^i S_t^i = h^T S_t,$$

where the super-index T denotes transpose. The normalized value process V^Z *corresponding to a portfolio h is defined as*

$$V^Z(t) = \sum_{i=0}^{N} h^i Z_t^i = h^T Z_t = \frac{V^S(t)}{S_t^0}.$$

An arbitrage portfolio *is a portfolio h such that*

$$\begin{aligned} V^S(0) &= 0, \\ V^S(1) &\geq 0 \text{ with probability 1} \\ E^P[V^S(1)] &> 0 \qquad (\text{or equivalently } P(V^S(1) > 0) > 0) \end{aligned}$$

or, given the assumptions on S^0, equivalently

$$\begin{aligned} V^Z(0) &= 0, \\ V^Z(1) &\geq 0 \text{ with probability 1} \\ E^P[V^Z(1)] &> 0. \end{aligned}$$

We are now ready to state our first version of the first fundamental theorem.

Proposition 1 *The market is free of arbitrage if and only if there exists a probability measure Q on Ω such that*

- $Q(\omega) > 0$ *for all* $\omega \in \Omega$
- $\dfrac{S_0^i}{S_0^0} = E^Q\left[\dfrac{S_1^i}{S_1^0}\right] \qquad i = 1, \ldots, N.$

Such a measure Q is called a martingale measure.

Remark 2 The first condition on Q means that P and Q are equivalent, which means that

$$P(A) = 0 \quad \Longleftrightarrow \quad Q(A) = 0,$$

or equivalently

$$P(A) = 1 \quad \Longleftrightarrow \quad Q(A) = 1,$$

The measures P and Q thus agree on what happens with probability one and what happens with probability zero.

The second condition means that the normalized price process Z_t is a martingale under Q.

Note that different numeraire assets will give rise to different martingale measures.

Also note that the larger M is the easier it is to find a martingale measure Q.

To do the proof we will use Farkas' Lemma, which is a special case of the separation theorem for convex sets.

Lemma 1 (Farkas' Lemma) *Given an $n \times m$-matrix A and an $n \times 1$-vector g, exactly one of the following systems can be solved*

$$E1 : \left\{ \begin{array}{rcl} A\lambda &=& g \\ \lambda &\geq& 0 \end{array} \right. \qquad E2 : \left\{ \begin{array}{rcl} g^T x &<& 0 \\ A^T x &\geq& 0 \end{array} \right.$$

Remark 3 The system $E2$ can be written as

$$E2 : \begin{cases} x^T g & < & 0 \\ x^T A & \geq & 0 \end{cases}$$

Proof: (of Proposition 1) Let the $(N+1) \times M$ matrix D be given by

$$D = \begin{bmatrix} Z_1^0(\omega_1) & Z_1^0(\omega_2) & \dots & Z_1^0(\omega_M) \\ Z_1^1(\omega_1) & Z_1^1(\omega_2) & \dots & Z_1^1(\omega_M) \\ \vdots & \vdots & & \vdots \\ Z_1^N(\omega_1) & Z_1^N(\omega_2) & \dots & Z_1^N(\omega_M) \end{bmatrix}$$

The first column of the matrix D is thus the normalized price vector at time $t = 1$ for the outcome ω_1, $Z_1(\omega_1)$. The first row of D is just a row of ones (recall that $Z_1^0 \equiv 1$).
We want to write the definition of an arbitrage portfolio as the system E1 or E2, in order to see what the other system looks like. The condition $V^Z(0) = 0$ can be written as

$$V_0^Z \geq 0, \quad \text{and} \quad V_0^Z \leq 0.$$

Using that $V_0^Z = h^T Z_0$ we get

$$h^T Z_0 \geq 0, \quad \text{and} \quad h^T Z_0 \leq 0,$$

or

$$h^T Z_0 \geq 0, \quad \text{and} \quad - h^T Z_0 \geq 0.$$

The condition $V^Z(1) \geq 0$ can be written as

$$h^T D \geq 0$$

and finally the condition $E^P[V^Z(1)] > 0$ can be written as

$$-h^T D p < 0$$

where $p = (P(\omega_1), \dots, P(\omega_M))^T$.
Let h play the role of x. Then there will exist no arbitrage portfolios if we can not solve the system E2 with

$$g = -Dp \quad \text{and} \quad A = [Z_0 \ \ -Z_0 \ \ D].$$

A is thus an $(N+1) \times (M+2)$-matrix.
So no arbitrage is equivalent to not being able to solve the system $E2$ with the given matrices. According to Farkas' lemma this means that no arbitrage it is equivalent to being able to solve the system $E1$ with the given matrices, i.e.

$$[Z_0 \ \ -Z_0 \ \ D]\lambda = -Dp \qquad \lambda \geq 0$$

or

$$Z_0(\lambda_2 - \lambda_1) = D(p + \lambda^*)$$

where $\lambda^* = (\lambda_3, \dots, \lambda_{M+2})$ (with $\lambda \geq 0$). The first row reads

$$\lambda_2 - \lambda_1 = \sum_{i=1}^{M} (p_i + \lambda_i^*)$$

or

$$\frac{1}{\lambda_2 - \lambda_1} \sum_{i=1}^{M} (p_i + \lambda_i^*) = 1$$

If we let

$$Q(\omega_i) = q_i = \frac{1}{\lambda_2 - \lambda_1}(p_i + \lambda_i^*)$$

we are done, since then $q_i > 0$, $i = 1, \ldots, M$ and

$$Z_0 = Dq = E^Q[Z_1],$$

where $q = (Q(\omega_1), \ldots, Q(\omega_M))^T$. □

We now formally define an (equivalent) martingale measure and restate Proposition 1. First recall from Remark 2 the definition of equivalent measures.

Definition 3 *Two probability measures P and Q on $(\Omega, \mathcal{F})$ are* equivalent *if*

$$P(A) = 0 \quad \Longleftrightarrow \quad Q(A) = 0 \quad \textit{for all } A \in \mathcal{F}.$$

If P and Q are equivalent we write $P \sim Q$.

Definition 4 *Consider the market above and fix the asset S^0 as the numeraire asset. A probability measure Q on $(\Omega, \mathcal{F})$ is said to be a* martingale measure *if*

1. *$Q \sim P$.*
2. *For every $i = 1, \ldots, N$ the normalized asset price process*

$$Z_t^i = \frac{S_t^i}{S_t^0},$$

   is a martingale under Q.

Theorem 1 (First Fundamental Theorem) *Given a fixed numeraire, the market is free of arbitrage if and only if there exists a martingale measure.*

1.3 Completeness

We continue by studying completeness of the market, i.e. if all derivatives are reachable. The formal definitions are given below.

Definition 5 *A* financial derivative *or* contingent claim *is any random variable defined on $(\Omega, \mathcal{F})$.*
A claim X is said to be reachable *if there is a portfolio h such that*

$$V_1^S = X, \textit{ with probability 1}$$

The portfolio h is said to be a hedging *or* replicating portfolio *for the claim X.*

Introduce the following $M \times (N+1)$-matrix A:

$$A = \begin{bmatrix} S_1^0(\omega_1) & S_1^1(\omega_1) & \dots & S_1^N(\omega_1) \\ S_1^0(\omega_2) & S_1^1(\omega_2) & \dots & S_1^N(\omega_2) \\ \vdots & \vdots & & \vdots \\ S_1^0(\omega_M) & S_1^1(\omega_M) & \dots & S_1^N(\omega_M) \end{bmatrix}$$

The rows of A is the price vector S_1 for the different outcomes ω_i, $i = 1, \dots, M$. The market is complete if we can find a portfolio h such that

$$V_1^S = X, \text{ with probability } 1,$$

which can be written more explicitly as

$$Ah = \begin{pmatrix} V_1^S(\omega_1) \\ V_1^S(\omega_2) \\ \vdots \\ V_1^S(\omega_M) \end{pmatrix} = \begin{pmatrix} X(\omega_1) \\ X(\omega_2) \\ \vdots \\ X(\omega_M) \end{pmatrix} \tag{1}$$

For the market to be complete we need to be able to solve the above system for all claims X (since then all claims will be reachable).

Proposition 2 *The market is complete if and only if A has rank M.*

Remark 4 Note that for the rank of A to be M we have to have that $M \leq N+1$, and that the larger N is, the easier it is to solve (1).

Also note that completeness is of no interest if the market is not free of arbitrage.

We are now ready to state the second fundamental theorem.

Theorem 2 (Second Fundamental Theorem) *Assume that the market is free of arbitrage, then the market is complete if and only if the martingale measure is unique.*

Remark 5 There are 0,1 or ∞ many martingale measures.

Proof: From Proposition 2 we know that the market is complete if and only if

$$Im[A] = \mathbb{R}^M.$$

Since $A = diag(S_1^0)D^T$ where $diag(S_1^0)$ denotes a diagonal matrix with the vector $S_1^0 = (S_1^0(\omega_1), \dots, S_1^0(\omega_M))$ on the diagonal we see that the market is complete if and only if

$$Im[D^T] = \mathbb{R}^M.$$

There exists a martingale measure if we can solve

$$\begin{aligned} Dq &= Z_0 \\ q &> 0. \end{aligned}$$

The solution is unique if and only if the kernel (or null space) of D is trivial, i.e.

$$Ker(D) = 0.$$

We now have the following duality result

$$(Im[D^T])^\perp = Ker(D).$$

So $Ker(D) = 0$ if and only if $Im[D^T] = \mathbb{R}^M$, i.e. the market is complete if and only if the martingale measure is unique. $\square$

1.4 Pricing

In this section we will only be looking at the value process associated with the unnormalized economy, so here super-index h will be used to indicate which portfolio the value process is associated with. We will do everything for a *fixed numeraire* asset S^0. Assume that the market is free of arbitrage and fix a claim X. If X is reachable by the replicating portfolio h we have for the price of X at time $t = 0$

$$\Pi(0; X) = V^h(0) = \sum_{i=0}^{N} h^i S_0^i.$$

Now, by the definition of a martingale measure, we have

$$S_0^i = S_0^0 E^Q \left[\frac{S_1^i}{S_1^0} \right].$$

Inserting this into the price of X we get

$$\Pi(0; X) = \sum_{i=0}^{N} h^i S_0^i = \sum_{i=0}^{N} h^i S_0^0 E^Q \left[\frac{S_1^i}{S_1^0} \right].$$

Now since h is deterministic we can move it into the expectation, and since expectation is linear we can move the sum into the expectation to obtain

$$\Pi(0; X) = \sum_{i=0}^{N} h^i S_0^0 E^Q \left[\frac{S_1^i}{S_1^0} \right] = S_0^0 E^Q \left[\sum_{i=0}^{N} h^i \frac{S_1^i}{S_1^0} \right].$$

Finally using the definition of the value process, and that h replicates X we obtain

$$\Pi(0; X) = S_0^0 E^Q \left[\sum_{i=0}^{N} h^i \frac{S_1^i}{S_1^0} \right] = S_0^0 E^Q \left[\frac{V_1^h}{S_1^0} \right] = S_0^0 E^Q \left[\frac{X}{S_1^0} \right] = S_0^0 E^Q \left[\frac{\Pi(1; X)}{S_1^0} \right].$$

Thus, the normalized price of the claim is also a martingale under Q!

Remark 6 If both h and g replicate X then

$$V^h(0) = V^g(0)$$

or there would be arbitrage.

Since

$$\Pi(0; X) = V^h(0) = S_0^0 E^Q \left[\frac{X}{S_1^0} \right].$$

the price of a reachable claim does not depend on which martingale measure we use (given that there are several for the fixed numeraire).

To sum up:

Proposition 3 *Suppose that the market is free of arbitrage. Then the following hold:*

1. *To extend the market with X and keep it free of arbitrage we have to set*

$$\Pi(0; X) = S_0^0 E^Q \left[\frac{X}{S_1^0} \right]. \tag{2}$$

   for some martingale measure Q.

2. *If X is not reachable different choices of martingale measures* for the fixed numeraire *give different prices.*

3. *If X is reachable (2) will not depend on the choice of martingale measure Q and*

$$\Pi(0; X) = V^h(0) \qquad \textit{for all } h \textit{ replicating } X.$$

For more on pricing in incomplete markets see for instance "Introduction to Mathematical Finance. Discrete Time Models", by Stanley R. Pliska.