1 Forwards and Futures

1.1 The model setup

Consider a discrete time model, so $t \in \{0, 1, \ldots, T\}$. Assume that there is a risk less asset with price process B, and a risky asset with price process S, possibly paying dividends. Dividend payments are specified by the cumulative dividend process D. The model is thus given by

$$\begin{bmatrix} B(t) & 0 \\ S(t) & D(t) \end{bmatrix}.$$

Remark 1 Note that when it comes to actually computing forwards and futures prices there is a difference between forwards and futures written on investment assets, which do not provide utility beyond monetary payoffs, and forwards and futures written on consumption assets, which do provide utility in other ways than monetary payoffs. Examples of investment assets are stocks, bonds, derivatives, and currencies. Examples of consumption assets are oil, meat and corn. We will mostly be dealing with investment assets.

1.2 Forwards

We start off by recalling what we know. Fix a martingale measure Q. Let X be a T-claim. We then know that

$$\Pi_t(X) = E^Q\left[B_t \frac{X}{B_T} \middle| \mathcal{F}_t \right].$$

The payment streams for the buyer/holder of X are that at time t the (spot) price $\Pi_t(X)$ is paid and at time T the payoff X is received.
If we instead consider a *forward contract* written on X, the payment streams to the buyer/holder of the forward contract are that at time t no transactions are made, but the *forward price* $f(t;T,X)$ is determined, and at time T the forward price $f(t;T,X)$ is paid and the payoff X is received.
Note that the forward contract can be seen as a T-claim Z where

$$Z = X - f(t;T,X),$$

and where $f(t;T,X)$ is determined in such a way that the (spot) price at time t, $\Pi_t(Z)$, is 0!

Definition 1 *Let X be a T-claim. By the* forward price process *we mean an $\underline{\mathcal{F}}$-adapted process $f(t;T,X)$ (thus $f(t;T,X) \in \mathcal{F}_t$) such that*

$$\Pi_t[X - f(t;T,X)] = 0.$$

The forward price can now be computed. We obtain

$$\begin{aligned} 0 &= \Pi_t[X - f(t;T,X)] \\ &= B_t E^Q\left[\frac{X - f(t;T,X)}{B_T} \middle| \mathcal{F}_t \right] \\ &= B_t E^Q\left[\frac{X}{B_T} \middle| \mathcal{F}_t \right] - B_t f(t;T,X) E^Q\left[\frac{1}{B_T} \middle| \mathcal{F}_t \right] \\ &= \Pi_t[X] - f(t;T,X) p(t,T). \end{aligned}$$

Here we have used that conditional expectation is linear and that $f(t;T,X) \in \mathcal{F}_t$ to obtain the third equality, and introduced

$$p(t,T) = B_t E^Q \left[\left. \frac{1}{B_T} \right| \mathcal{F}_t \right],$$

as the price at time t of a contract which pays \$1 at time T, a so called *zero coupon bond*. We have thus proved:

Proposition 1 *The forward price process is given by*

$$f(t;T,X) = \frac{\Pi_t[X]}{p(t,T)}.$$

Example 1 If $X = S_T$ where S is the price process of a stock which does not pay dividends the forward price is

$$f(t;T,X) = \frac{\Pi_t[S_T]}{p(t,T)} = \frac{S_t}{p(t,T)}.$$

If furthermore $B_t = e^{rt}$, where r is constant, then

$$f(t;T,X) = \frac{S_t}{p(t,T)} = e^{r(T-t)} S_t.$$

□

Example 2 If $X = S_T$ where S is the price process of a stock with dividend process D we obtain

$$f(t;T,X) = \frac{\Pi_t[S_T]}{p(t,T)} = \frac{B_t E^Q[S_T/B_T|\mathcal{F}_t]}{p(t,T)}.$$

Recall that for dividend paying assets G^Z is a Q-martingale which resulted in the formula

$$S_t = E^Q \left[B_t \frac{S_T}{B_T} + \sum_{n=t+1}^{T} \frac{B_t}{B_n} \Delta D_n \right| \left. \mathcal{F}_t \right]$$

This yields that

$$B_t E^Q \left[\left. \frac{S_T}{B_T} \right| \mathcal{F}_t \right] = S_t - E^Q \left[\left. \sum_{n=t+1}^{T} \frac{B_t}{B_n} \Delta D_n \right| \mathcal{F}_t \right]$$

and therefore

$$f(t;T,X) = \frac{1}{p(t,T)} \left\{ S_t - E^Q \left[\left. \sum_{n=t+1}^{T} \frac{B_t}{B_n} \Delta D_n \right| \mathcal{F}_t \right] \right\}$$

If $B_t = e^{rt}$ we obtain

$$f(t;T,X) = e^{r(T-t)} S_t - E^Q \left[\left. \sum_{n=t+1}^{T} e^{r(T-n)} \Delta D_n \right| \mathcal{F}_t \right].$$

□

Remark 2 Fix t, T and X and let $t \leq u \leq T$. It is important to distinguish between the following two prices

1. The forward price $f(u;T,X)$ which is to be paid at time T to the underwriter of a forward contract contracted at time u.
2. The (spot) price at time u of a forward contract contracted at time t (with time of delivery T). This price is given by

$$\Pi_u(X - f(t;T,X)) = \Pi_u(X) - p(u,T)f(t;T,X).$$

An alternative way of computing the forward price is given by the following proposition.

Proposition 2 *The forward price process is given by*

$$f(t;T,X) = E^T[X|\mathcal{F}_t],$$

where the super index T indicates that the expectation should be taken with respect to the T-forward measure Q^T, which is the martingale measure which has the zero coupon bond with maturity T as numeraire.

Proof: Let $p_t^T = p(t,T)$. Fix a T-claim X. Under Q^T we know that

$$\frac{\Pi_t(X)}{p_t^T}$$

should be a martingale. This means that

$$\frac{\Pi_t(X)}{p_t^T} = E^T\left[\left.\frac{\Pi_T(X)}{p_T^T}\right|\mathcal{F}_t\right] = E^T[X|\mathcal{F}_t]$$

where we have used that $\Pi_T(X) = X$, and that $p_T^T = 1$. Using the above formula, and Proposition 1 we get

$$f(t;T,X) = \frac{\Pi_t[X]}{p(t,T)} = E^T[X|\mathcal{F}_t].$$

$\square$

1.3 Futures

Now let us consider a *futures contract* written on X. The payment streams to the buyer/holder of the futures contract are that at each point in time t after the contract has been entered the change in the *futures price* $\Delta F(t;T,X)$ (may be negative!) is received and at time T the futures price $F(T;T,X)$ is paid and the payoff X is received. Note that the spot price of a futures contract is always zero.
In order for the spot price of the contract to be zero at time T, we have to have

$$F(T;T,X) = X.$$

Because of this many futures contracts are terminated before T.

The futures contract is characterized by the payment

$$\Delta F(t;T,X) = F(t;T,X) - F(t-1;T,X)$$

at time t, which is known as "marking to market".
Since the spot price is always zero there is no cost associated with entering or closing a futures contract.
For the practical details surrounding a futures contract, see "Fundamentals of Futures and Options Markets" by John C. Hull.
Formally we will define a futures contract as follows:

Definition 2 *Let X be a T-claim. A futures contract written on X with delivery time T is an asset with adapted price and dividend processes $[\Pi, D]$ such that*

1. $D(t) = F(t;T,X)$,
2. $F(T;T,X) = X$,
3. $\Pi_t = 0$ *for all* $t \leq T$.

We now have the following proposition regarding the futures price process.

Proposition 3 *Assume that the price process of the risk less asset is predictable. Then the futures price process $F(t;T,X)$ is given by*

$$F(t;T,X) = E^Q[X|\mathcal{F}_t], \qquad t \leq T.$$

Proof: We know that the normalized gain process associated with the futures contract G^Z is a Q-martingale, where

$$G_t^Z = \frac{\Pi_t}{B_t} + \sum_{n=1}^{t} \frac{1}{B_n} \Delta F_n(T,X)$$

Above we have used the notation $F(t;T,X) = F_t(T,X)$. Since the spot price Π_t of a futures contract is zero we have

$$\Delta G_t^Z = \frac{1}{B_t} \Delta F_t(T,X).$$

Using that G^Z is a Q-martingale we get

$$0 = E^Q[\Delta G_t^Z|\mathcal{F}_{t-1}] = E^Q\left[\frac{1}{B_t}\Delta F_t(T,X) \middle| \mathcal{F}_{t-1}\right].$$

If B_t is predictable, then $B_t \in \mathcal{F}_{t-1}$ and we can take it out of the conditional expectation to obtain

$$E^Q\left[\Delta F_t(T,X)| \mathcal{F}_{t-1}\right] = 0.$$

This means that the futures pricess process is a Q-martingale! The martingale property can also be written

$$F_t(T,X) = E^Q\left[F_T(T,X)|\mathcal{F}_t\right] \qquad \text{for } t \leq T.$$

The result will now follow if we insert the boundary value $F_T(T,X) = X$. $\square$

1.3.1 Using a binomial tree for options on futures contracts

Recall that the futures price process is a martingale under Q. This means that when we build the tree we should make sure that

$$f = E^Q[F_{t+1}|F_t = f]$$

or, more explicitly

$$f = q \cdot fu + (1-q) \cdot fd.$$

For a concrete example of the computations, see Example 12.3 in "Fundamentals of Futures and Options Markets" and Figure 12.13 in "Options, Futures and Other Derivatives".

1.4 When are forward and futures prices equal?

In general forward and futures prices are **not** equal. Under certain assumptions they are though.

Lemma 1 *Fix a T-claim X. If interest rates are deterministic then*

$$f(t;T,X) = F(t;T,X) = E^Q[X|\mathcal{F}_t].$$

Proof: We only do the proof for the case of a constant interest rate. If $B_t = e^{rt}$, where r is a constant, we have that

$$\begin{aligned} f(t;T,X) &= \frac{\Pi_t[X]}{p(t,T)} = \frac{e^{-r(T-t)}E^Q[X|\mathcal{F}_t]}{e^{-r(T-t)}} \\ &= E^Q[X|\mathcal{F}_t] = F(t;T,X). \end{aligned}$$

$\square$

Example 3 If $B_t = e^{rt}$, where r is a constant, and dividends are assumed to be deterministic, we can give an arbitrage argument for the formula

$$f(t;T,X) = e^{r(T-t)}S_t - E^Q\left[\left.\sum_{n=t+1}^{T} e^{r(T-n)}\Delta D_n\right| \mathcal{F}_t\right]. \tag{1}$$

First note that if $B_t = e^{rt}$ then

$$p(t,T) = B_t E^Q\left[\left.\frac{1}{B_T}\right| \mathcal{F}_t\right] = e^{-r(T-t)}.$$

Now do the following.

- Buy the stock at time t at the price S_t.
- Sell $S_t/p(t,T)$ T-bonds at time t.

The net investment at time t is zero.

- Whenever the stock pays dividends invest these in bonds maturing at time T.

•At time T you will

$$\begin{array}{ll} \text{receive} & S_T \\ \text{pay} & \dfrac{S_t}{p(t,T)} \\ \text{receive} & \displaystyle\sum_{n=t+1}^{T} \frac{\Delta D_n}{p(n,T)} \end{array}$$

In total you will receive

$$S_T - \frac{S_t}{p(t,T)} + \sum_{n=t+1}^{T} \frac{\Delta D_n}{p(n,T)}.$$

A forward contract pays

$$S_T - f(t;T,S_T).$$

Since both strategies only result in payment at time T and both the forward price $f(t;T,S_T)$ and $\frac{S_t}{p(t,T)} + \sum_{n=t+1}^{T} \frac{\Delta D_n}{p(n,T)}$ are known at time t we must have

$$f(t;T,S_T) = \frac{S_t}{p(t,T)} - \sum_{n=t+1}^{T} \frac{\Delta D_n}{p(n,T)}.$$

Using the expression for $p(t,T)$ we obtain (1).

□