1 The Black-Scholes model

1.1 The model setup

In the simplest version of the Black-Scholes model the are two assets: a risk-less asset (bank account or bond) with price process $B(t)$ at time t, and a risky asset (stock) with price process $S(t)$ at time t. The price dynamics of the two price processes under the *objective probability measure* P are

$$\begin{cases} dB(t) &= rB(t)dt, \\ B(0) &= 1, \end{cases} \tag{1}$$

where r is a constant and

$$\begin{cases} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t), \\ S(0) &= s_0, \end{cases} \tag{2}$$

where μ and σ are constants and W is what is known as *Wiener process*. Defining the local rate of return of the risk-less asset B as

$$\frac{dB(t)}{B(t)dt} = r,$$

we see that it is deterministic, whereas the rate of return on the stock given by

$$\frac{dS(t)}{S(t)dt} = \mu + \sigma \frac{dW(t)}{dt},$$

is stochastic, since there is the "white noise" term $dW(t)/dt$. We see that μ represents the local mean rate of return of the stock. The constant σ is known as the *volatility* of the stock and determines how much influence the random term dW has in (2).
Both Equations (1) and (2) are explicitly solvable and the solutions are

$$B(t) = e^{rt}, \qquad S(t) = s_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \right\}.$$

The solution for the risk-less asset offers no problems, but what can we say about the solution to the stock price equation? Well, the distribution of the exponent is normal with expectation

$$E\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] = \left(\mu - \frac{1}{2}\sigma^2\right) t,$$

and variance

$$V\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] = \sigma^2 t.$$

The distribution of the stock price *under the objective measure* P is therefore log-normal with the parameters given above.

1.2 The Black-Scholes formula

Just as for the binomial model the model will be free of arbitrage if there exists a martingale measure, and complete if the martingale measure is unique. The definition of a martingale measure is basically the same as before:

Definition 1 *Consider the market above and fix the asset B as the numeraire asset. A probability measure Q is said to be a* (risk neutral) martingale measure *if*

1. *$Q \sim P$.*
2. *The normalized stock price process*

$$Z_t = \frac{S_t}{B_t},$$

is a martingale under Q.

It turns out that there is a unique martingale measure Q for the Black-Scholes market, which means that the market is free of arbitrage and complete. Under Q the local mean rate of return of the stock has to be equal to the local rate of return of the risk-less asset. This means that under Q the dynamics of the stock price process will look as follows

$$\begin{cases} dS(t) &= rS(t)dt + \sigma S(t)d\widehat{W}(t), \\ S(0) &= s_0, \end{cases} \tag{3}$$

with the solution

$$S(t) = s_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\widehat{W}_t\right\}.$$

Here $\widehat{W}$ is a Q-Wiener process.

Now, suppose that we want to compute the price of a European call option with strike price K and expiration date T. Just as before it turns out that to ensure that there is no arbitrage on the market we should make sure that all normalized price processes are Q-martingales. This means that we get the following pricing formula:

Proposition 1 *Let $X = \phi(S_T)$ be a (simple) contingent T-claim. The price of X at time t, $\Pi_t(X)$, is given by*

$$\Pi_t(X) = e^{-r(T-t)}E^Q[\phi(S_T)|\mathcal{F}_t]. \tag{4}$$

Proof: We want

$$\frac{\Pi_t(X)}{B_t}$$

to be a Q-martingale. This means that we should have

$$\frac{\Pi_t(X)}{B_t} = E^Q\left[\left.\frac{\Pi_T(X)}{B_T}\right| \mathcal{F}_t\right].$$

Using that $B_t = e^{rt}$ and that $\Pi_T(X) = X$ for arbitrage reasons, we obtain the pricing formula (4) in the proposition. $\square$

For a European call option we have

$$X = \max\{S_T - K, 0\}$$

and the price at time zero is therefore

$$\Pi_0(X) = e^{-rT}E^Q[\max\{S_T - K, 0\}].$$

where

$$S(T) = s_0 e^Z$$

and Z is normally distributed with with expectation $m = \left(r - \frac{1}{2}\sigma^2\right) T$ and standard deviation $s = \sigma\sqrt{T}$. Let φ denote the density function of the distribution of Z. Then we have

$$\begin{aligned}
E^Q[\max\{S_T - K, 0\}] &= E^Q[\max\{s_0 e^Z - K, 0\}] \\
&= 0 \cdot Q(s_0 e^Z \le K) + \int_{\ln(K/s_0)}^{\infty} (s_0 e^z - K)\,\varphi(z) dz \\
&= s_0 \int_{\ln(K/s_0)}^{\infty} e^z \frac{1}{\sqrt{2\pi s^2}} e^{-(z-m)^2/(2s^2)} dz - KQ\left(Z > \ln\left(\frac{K}{s_0}\right)\right).
\end{aligned}$$

By completing the square in the exponent of the integrand we see that the first term can be written as

$$\begin{aligned}
& s_0 \int_{\ln(K/s_0)}^{\infty} e^z \frac{1}{\sqrt{2\pi s^2}} e^{-(z-m)^2/(2s^2)} dz \\
=\;& s_0 e^{s^2/2+m} \int_{\ln(K/s_0)}^{\infty} \frac{1}{\sqrt{2\pi s^2}} e^{-(z-(m+s^2))^2/(2s^2)} dz \\
=\;& s_0 e^{s^2/2+m} Q(Y > \ln\left(\frac{K}{s_0}\right)),
\end{aligned}$$

where Y is normally distributed with expectation $m + s^2$ and standard deviation s. Now use the following relations:

1. For any stochastic variable X: $Q(X > x) = 1 - Q(X \le x)$.
2. If X is normally distributed with expectation a and standard deviation b we have

   $$Q(X \le x) = N\left(\frac{x-a}{b}\right),$$

   where N denotes the cumulative distribution function of the standard normal distribution.
3. $1 - N(x) = N(-x)$.

You will the arrive at the Black-Scholes formula.

Proposition 2 (Black-Scholes formula) *The price of a European call option at time t with strike price K and expiry date T is given by $c(t, S_t)$, where*

$$c(t,s) = sN[d_1(t,s)] - e^{-r(T-t)} K N[d_2(t,s)].$$

Here

$$\begin{cases}
d_1(t,s) &= \dfrac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\dfrac{s}{K}\right) + \left(r + \dfrac{1}{2}\sigma^2\right)(T-t)\right\}, \\
d_2(t,s) &= d_1 - \sigma\sqrt{T-t}.
\end{cases}$$

and N denotes the cumulative distribution function of the standard normal distribution.

Remark 1 Note that you can clearly see from the above proposition that arbitrage pricing is pricing in terms of the underlying asset. Given that we know the stock price today we can compute the price of the option, otherwise we are at a loss.

1.3 The Black-Scholes formula as the limit of the binomial model formula

The objective of this section is to show that if a binomial model with carefully chosen parameters is used to compute the price of a European call option, the price will converge to the Black-Scholes formula as we let the number of time steps in the binomial tree go to infinity. Consider pricing a European call option with strike price K and exercise time T on a stock with a yearly volatility of σ. Divide each year into n periods. This gives a binomial model with nT periods. In this tree, which we label the nth tree, we choose the following parameters

$$\begin{aligned} u_n &= \exp\left\{\sigma\sqrt{\frac{1}{n}}\right\}, \\ d_n &= \exp\left\{-\sigma\sqrt{\frac{1}{n}}\right\} = \frac{1}{u_n}, \\ 1+R_n &= \exp\left\{\frac{r}{n}\right\}. \end{aligned}$$

Here r denotes the risk-free interest rate per annum with continuous compounding and R_n denotes the one period interest rate in the binomial model.

From Proposition 4 in lecture 4 we obtain that the price at time $t=0$ of the option computed in the nth model, c^n, is given by

$$c^n = \frac{1}{(1+R_n)^{nT}} \sum_{k=0}^{nT} \binom{nT}{k} q_n^k (1-q_n)^{nT-k} \max\{S_0 u_n^k d_n^{nT-k} - K, 0\},$$

where S_0 denotes the stock price at time $t=0$ and

$$q_n = \frac{(1+R_n)-d_n}{u_n-d_n}.$$

We will now proceed in a number of steps.

1. To start off with we use that $\max\{S-K,0\} = (S-K)\cdot I\{S>K\}$ where $I\{A\}$ denotes the indicator function of the set A, i.e.

$$I\{A\}(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{otherwise.} \end{cases}$$

to show that c^n can be written in the following way

$$\begin{aligned} c^n &= S_0 \sum_{k=0}^{nT} \binom{nT}{k} (q_n')^k (1-q_n')^{nT-k} I\{S_0 u_n^k d_n^{nT-k} > K\} \\ &\quad - \frac{K}{(1+R_n)^{nT}} \sum_{k=0}^{nT} \binom{nT}{k} q_n^k (1-q_n)^{nT-k} I\{S_0 u_n^k d_n^{nT-k} > K\}, \end{aligned}$$

where

$$q_n' = \frac{u_n}{1+R_n} q_n.$$

First

$$\begin{aligned} c^n &= \frac{1}{(1+R_n)^{nT}} \sum_{k=0}^{nT} \binom{nT}{k} (q_n)^k (1-q_n)^{nT-k} S_0 u_n^k d_n^{nT-k} I\{S_0 u_n^k d_n^{nT-k} > K\} \\ &\quad - \frac{1}{(1+R_n)^{nT}} \sum_{k=0}^{nT} \binom{nT}{k} q_n^k (1-q_n)^{nT-k} K I\{S_0 u_n^k d_n^{nT-k} > K\}, \end{aligned}$$

or

$$
\begin{aligned}
c^n &= S_0 \sum_{k=0}^{nT} \begin{pmatrix} nT \\ k \end{pmatrix} \left(\frac{q_n u_n}{1+R_n}\right)^k \left(\frac{(1-q_n)d_n}{1+R_n}\right)^{nT-k} I\{S_0 u_n^k d_n^{nT-k} > K\} \\
&\quad - \frac{K}{(1+R_n)^{nT}} \sum_{k=0}^{nT} \begin{pmatrix} nT \\ k \end{pmatrix} q_n^k (1-q_n)^{nT-k} I\{S_0 u_n^k d_n^{nT-k} > K\}.
\end{aligned}
$$

Since

$$1 - q_n' = 1 - \frac{u_n}{1+R_n} q_n = \frac{1+R_n - u_n q_n}{1+R_n}$$

and the definition of q_n tells us that

$$u_n q_n = 1 + R_n - d_n + d_n q_n,$$

we have that

$$1 - q_n' = \frac{1+R_n - (1+R_n - d_n + d_n q_n)}{1+R_n} = \frac{d_n(1-q_n)}{1+R_n}$$

and thus

$$
\begin{aligned}
c^n &= S_0 \sum_{k=0}^{nT} \begin{pmatrix} nT \\ k \end{pmatrix} (q_n')^k (1-q_n')^{nT-k} I\{S_0 u_n^k d_n^{nT-k} > K\} \\
&\quad - \frac{K}{(1+R_n)^{nT}} \sum_{k=0}^{nT} \begin{pmatrix} nT \\ k \end{pmatrix} q_n^k (1-q_n)^{nT-k} I\{S_0 u_n^k d_n^{nT-k} > K\},
\end{aligned}
$$

where

$$q_n' = \frac{u_n}{1+R_n} q_n.$$

2. Now use that

$$E^P[I\{A\}] = P(A)$$

to see that

$$c^n = S_0 Q'(\ln S_n(T) > \ln K) - Ke^{-rT} Q(\ln S_n(T) > \ln K),$$

where $S_n(T) = S_0 u_n^Y d_n^{Tn-Y}$ and $Y \in Bin(nT, q_n)$ under Q and $Y \in Bin(nT, q_n')$ under Q'.

Using that

$$E^P[I\{A\}] = P(A),$$

and the definition of R_n we readily see that

$$c^n = S_0 Q'(S_n(T) > K) - Ke^{-rT} Q(S_n(T) > K),$$

where $S_n(T) = S_0 u_n^Y d_n^{Tn-Y}$ and $Y \in Bin(nT, q_n)$ under Q, and $Y \in Bin(nT, q_n')$ under Q'. Since the logarithm is a strictly increasing function we have that

$$Q(S_n(T) > K) = Q(\ln S_n(T) > \ln K),$$

and thus

$$c^n = S_0 Q'(\ln S_n(T) > \ln K) - Ke^{-rT} Q(\ln S_n(T) > \ln K).$$

3. It should come as no surprise that the Central Limit Theorem will come in to play eventually. In order to use it we will need to compute the following quantities

$$\begin{aligned} M_n^Q &= E^Q[\ln S_n(T)], & V_n^Q &= Var^Q[\ln S_n(T)], \\ M_n^{Q'} &= E^{Q'}[\ln S_n(T)], & V_n^{Q'} &= Var^{Q'}[\ln S_n(T)]. \end{aligned}$$

What we really need to know is what these quantities tend to as n tends to infinity. Straight forward calculations give

$$\begin{aligned} M_n^Q &= E^Q[\ln S_n(T)] = E^Q[\ln S_0 + Tn \ln d_n + Y(\ln u_n - \ln d_n)] \\ &= \ln S_0 + Tn(q_n \ln u_n + (1-q_n)\ln d_n), \\ M_n^{Q'} &= E^{Q'}[\ln S_n(T)] = \ln S_0 + Tn(q'_n \ln u_n + (1-q'_n)\ln d_n), \\ V_n^Q &= Var^Q[\ln S_n(T)] = Var^Q[\ln S_0 + Tn \ln d_n + Y(\ln u_n - \ln d_n)] \\ &= Tnq_n(1-q_n)(\ln u_n - \ln d_n)^2, \\ V_n^{Q'} &= Var^{Q'}[\ln S_n(T)] = Tnq'_n(1-q'_n)(\ln u_n - \ln d_n)^2. \end{aligned}$$

Now rewrite the expression for M_n^Q using the definitions of u_n, d_n, R_n, and q_n.

$$\begin{aligned} M_n^Q - \ln S_0 &= Tn\left(\frac{\sigma}{\sqrt{n}}\frac{e^{r/n} - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} - \frac{\sigma}{\sqrt{n}}\frac{e^{\sigma/\sqrt{n}} - e^{r/n}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right) \\ &= T\sqrt{n}\sigma\left(\frac{2e^{r/n} - e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right). \end{aligned}$$

The MacLaurin expansion of the second order of e^x reads $e^x = 1 + x + x^2/2 + \mathcal{O}(x^3)$ so

$$\begin{aligned} e^{r/n} &= 1 + r/n + \mathcal{O}(1/n^2), \\ e^{\pm\sigma/\sqrt{n}} &= 1 \pm \sigma/\sqrt{n} + \sigma^2/(2n) + \mathcal{O}(1/n^{3/2}). \end{aligned}$$

Inserting this into the expression for M_n^Q yields

$$\begin{aligned} M_n^Q - \ln S_0 &= T\sqrt{n}\sigma\left(\frac{2r/n - \sigma^2/n + \mathcal{O}(1/n^{3/2})}{2\sigma/\sqrt{n} + \mathcal{O}(1/n^{3/2})}\right), \\ &= T\sigma\left(\frac{2r - \sigma^2 + \mathcal{O}(1/\sqrt{n})}{2\sigma + \mathcal{O}(1/n)}\right) \\ &\to T\left(r - \frac{\sigma^2}{2}\right) \quad \text{as } n \to \infty. \end{aligned}$$

Similar calculations for V_n^Q, $M_n^{Q'}$, and $V_n^{Q'}$ give

$$\begin{aligned} M_n^{Q'} - \ln S_0 &\to T\left(r + \frac{\sigma^2}{2}\right), \\ V_n^Q &\to \sigma^2 T, \\ V_n^{Q'} &\to \sigma^2 T. \end{aligned}$$

4. Another way to think of $\ln S_n(T)$ is as a sum of nT independent variables with possible outcomes $(\ln u_n, \ln d_n)$ and associated probabilities q_n and $1-q_n$ (or q_n' and $1-q_n'$) (use that $\ln ab = \ln a + \ln b$ to see this!). This means that we have a sum of independent random variables for which the first and second moment converge. This puts you in a position to use a version of the Central Limit Theorem[1] to determine what the distribution of $\ln S_n(T)$ tends to as n tends to infinity. Use the result to compute

$$\lim_{n\to\infty} c^n = \lim_{n\to\infty} \{S_0 Q'(\ln S_n(T) > \ln K) - Ke^{-rT} Q(\ln S_n(T) > \ln K)\}.$$

What (a version of) the Central Limit Theorem tells us is that the limit of the sum is normally distributed, i.e.

$$\ln S_n(T) \overset{Q/Q'}{\to} N(\ln S_0 + (r \pm \sigma^2/2)T, \sigma^2 T).$$

Thus, we should substitute $X \in N(\ln S_0 + (r \pm \sigma^2/2)T, \sigma^2 T))$ for $\ln S_n(T)$ when computing the limits of the probabilities on the right hand side. We therefore obtain

$$\begin{aligned}
\lim_{n\to\infty} c^n &= \lim_{n\to\infty} \{S_0 Q'(\ln S_n(T) > \ln K) - Ke^{-rT} Q(\ln S_n(T) > \ln K)\} \\
&= S_0 Q'(X > \ln K) - Ke^{-rT} Q(X > \ln K) \\
&= S_0 Q' \left(\frac{X - \ln S_0 - (r+\sigma^2/2)T}{\sigma\sqrt{T}} > \frac{\ln K - \ln S_0 - (r+\sigma^2/2)T}{\sigma\sqrt{T}} \right) \\
&\quad - Ke^{-rT} Q \left(\frac{X - \ln S_0 - (r-\sigma^2/2)T}{\sigma\sqrt{T}} > \frac{\ln K - \ln S_0 - (r-\sigma^2/2)T}{\sigma\sqrt{T}} \right).
\end{aligned}$$

After some final rewriting (multiply both sides inside Q and Q' by -1, thus reversing the inequality, and use that $N(0,1)$-distributed variables are symmetric and continuous, and that $\ln(x/y) = \ln x - \ln y$) we obtain

$$\lim_{n\to\infty} c^n = S_0 N(d_1) - Ke^{-rT} N(d_2).$$

Here N denotes the distribution function of a standard normal distribution, and

$$\begin{cases} d_1 &= \dfrac{1}{\sigma\sqrt{T}} \left\{ \ln\left(\dfrac{S_0}{K}\right) + \left(r + \dfrac{1}{2}\sigma^2\right) T \right\}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{cases}$$

1.4 Parity relations

To actually replicate a claim usually requires continuous rebalancing of the portfolio. Problems then arise due to transaction costs (among other things). The question is now which contracts can be replicated using constant (buy-and-hold) portfolios?
To investigate this we will use that a simple T-claim in this model is a random variable of the form

$$X = \phi(S_T),$$

where the given function ϕ is known as the *contract function.*

[1] You cannot make do with the version you know from the basic course in probability theory since that version requires a scaled sum of *identically* distributed random variables. Not to worry though, the Lindeberg-Feller-version of the Central Limit Theorem will apply to the situation at hand.

Fix a maturity T. We consider using the following assets in our portfolio:

Asset	$T-$ Bond	Stock	Call option
Contract function	$\phi_B(x) = 1$	$\phi_S(x) = x$	$\phi_{c,K}(x) = \max\{x - K, 0\}$
Price at time t	$e^{-r(T-t)}$	S_t	$c_K(t, S_t)$

The call options used should be European and we will allow using several call options with different strike prices K (hence the sub-index K in the pricing function).
Fix a simple T-claim $X = \phi(S_T)$. If the contract function ϕ satisfies

$$\phi(x) = \alpha\phi_B(x) + \beta\phi_S(x) + \sum_{i=1}^{n} \gamma_i \phi_{c,K_i}(x),$$

for some real numbers α, β, γ_i, K_i, $i = 1, \ldots, n$, then the price of X can be obtained as

$$\Pi_t(X) = \alpha\Pi_t(\phi_B) + \beta\Pi_t(\phi_S) + \sum_{i=1}^{n} \gamma_i \Pi_t(\phi_{c,K_i}),$$

and X is replicated by the constant portfolio $(\alpha, \beta, \gamma_1, \ldots, \gamma_n)$. To prove it use the risk neutral valuation formula (4) and that the conditional expectation is linear.
Now consider a European put option with strike price K. This has a contract function $\phi_{p,K}$ given by

$$\phi_{p,K}(x) = \max\{K - x, 0\}.$$

If you draw a picture you will find that

$$\phi_{p,K}(x) = K\phi_B - \phi_S + \phi_{c,K}.$$

which gives us the following.

Proposition 3 (Put-call parity) *Consider a European call option and a European put option, both with strike price K and expiration date T. Denoting the pricing functions $c(t, s)$ and $p(t, s)$, respectively, we have the following relation*

$$p(t, S_t) = Ke^{-r(T-t)} + c(t, S_t) - S_t.$$

This means that we can replicate a put option by buying K zero coupon bonds with maturity T, and a call option with the same strike and maturity as the put option, and selling the underlying stock.
In fact one can show the following.

Proposition 4 *All continuous contract functions with compact support can be replicated with arbitrary precision (in sup-norm) using bonds, call options and the underlying stock.*

Remark 2 The problem with the above proposition is that in general the portfolio will contain a huge amount of assets. If you have a piecewise linear contract function you will do fine though!

1.5 Volatility

In order to use the Black-Scholes formula for pricing you need the values of t, T, r, σ, and S_t. Obtaining t, T (or rather $T - t$), and S_t does not present a problem, and proxies for r are available. That leaves us with the volatility σ. There are two basic approaches, namely to use "historic volatility" or "implied volatility".

1.5.1 Historic volatility

Using historic volatility means that we estimate σ using historical stock price data.
Suppose that we will observe the stock price at equidistant points in time $t_0, t_1, \ldots, t_n$, where $t_i - t_{i-1} = \Delta t$. Now let

$$Z_i = \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right), \qquad i = 1, \ldots, n.$$

Then $Z_1, \ldots, Z_n$ are independent, normally distributed random variables with

$$E[Z_i] = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t, \quad \text{and} \quad V(Z_i) = \sigma^2 \Delta t,$$

under P. Note that we can not make observations under Q!
From standard statistical theory we know that

$$S_Z = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2}, \text{ where } \bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$$

is an estimate of the standard deviation of Z_i, i.e. of $\sigma\sqrt{\Delta t}$. Therefore

$$\sigma^* = \frac{S_Z}{\sqrt{\Delta t}},$$

is an estimate of σ.
An estimate of the standard deviation of the estimate σ^* can be shown to be

$$D(\sigma^*) \approx \frac{\sigma^*}{\sqrt{2n}}.$$

An argument against using historical volatility is that for pricing purposes we want to know what will happen in the future, and this may not be reflected in historical data.

1.5.2 Implied volatility

We want to price in such a way that the prices we compute are consistent with the prices already observable in the market. This means that we should really use the market expectation of the volatility for the time to maturity of interest. We can achieve this by getting market data for a "benchmark" option written on the same stock as the option we want to value. For this option the price p should be known, and therefore we can back out σ from the relation

$$p = c(t, T, r, \sigma, K, S_t)$$

where $c(t, T, r, \sigma, K, S_t)$ is given by the Black-Scholes formula. The value of σ obtained in this way is called the *implied volatility*.

Implied volatilities can be used to test the Black-Scholes model. If we compute the implied volatility for European call options written on the same stock with the same maturity, but with different strike prices, and plot the implied volatility as a function of the strike price it should come out as a horizontal line (the Black-Scholes model assumes constant volatility!). Empirically it is often observed that options far out of the money or deep into the money are traded at higher volatilities, than options at the money. The graph of the implied volatilities then looks like a smile, and for this reason the implied volatility curve is termed the *volatility smile*.