1 The Black-Scholes model: extensions and hedging

1.1 Dividends

Since we are now in a continuous time framework the dividend paid out at time t (or $t-$) is given by $dD_t = D_t - D_{t-}$, where as before D denotes the cumulative dividend process of the risky asset.

1.1.1 Known dollar dividend

This means that for certain points in time between now, time t, and the option maturity, at time T, we have dividend payments of known dollar size. Let the dividend times be t_i, $i = 1, \ldots, n$, such that $t < t_1 < \ldots < t_n < T$ and the dividend payments be given by

$$dD_{t_i} = d_i.$$

Since the Black-Scholes model implies a lognormal distribution of the stock price, the stock price is guaranteed to always be positive, **but we can not guarantee it to be greater than any** $\epsilon > 0$. This means that the Black-Scholes model is not consistent with the type of dividend payments that has just been described.
As for the binomial model we will make an engineering fix and decompose the stock price into a risky part S^* and a risk less part which is the present value of future dividends $PV(div)$

$$S_t = S_t^* + PV(div) = S_t^* + \sum_{i=1}^{n} e^{-r(t_i - t)} d_i.$$

What determines the (European) option price is S_t^*, so the price of an option, on a stock which pays dividends of known dollar size, can be computed using the Black-Scholes formula with today's stock price S_t, replaced with

$$S_t^* = S_t - PV(div) = S_t - \sum_{i=1}^{n} e^{-r(t_i - t)} d_i$$

(recall that the holder of the option will not receive any dividend payments). See Example 13.6 in "Fundamentals of Futures and Options Markets", or Example 14.9 in "Options, Futures, and Other Derivatives" for a concrete example of the calculations.

1.1.2 Known dividend yield paid discretely

This means that for certain points in time between now, time t, and the option maturity, at time T, we have dividend payments which are a given fraction of the stock price at that time. Let the dividend times be t_i, $i = 1, \ldots, n$, such that $t < t_1 < \ldots < t_n < T$ and the dividend payments be given by

$$dD_{t_i} = \delta_i S_{t_i -}.$$

Note that these dividend payments can always be guaranteed, if the stock price is almost zero the dividend will be very small, that is all!
It can be shown that the price of an option, on a stock which gives a dividend yield paid discretely, can be computed using the Black-Scholes formula with today's stock price S_t, replaced with

$$S_t^y = \prod_{i=1}^{n} (1 - \delta_i) S_t.$$

Normally one would have $\delta_i = \delta$, $i = 1, 2, \ldots, n$.

1.1.3 Continuous dividend yield

This means that a constant fraction of the stock price is distributed as dividends continuously, i.e.

$$dD_t = \delta S_t dt.$$

Now recall that under the martingale measure Q all price processes or in the case with dividends present, gain processes have to have the same local mean rate of return as the risk-less asset, that is r. Let us use this to determine the dynamics of S under Q, when $dD_t = \delta S_t dt$. We have that $G_t = S_t + D_t$, which means that

$$\begin{aligned} dG_t &= dS_t + dD_t = \alpha S_t dt + \sigma S_t d\widehat{W}_t + \delta S_t dt \\ &= (\alpha + \delta) S_t dt + \sigma S_t d\widehat{W}_t. \end{aligned}$$

In order to have a local mean rate of return equal to r we have to set $\alpha = r - \delta$, thus the Q dynamics of a stock paying a continuous dividend yield have to be

$$dS_t = (r - \delta) S_t dt + \sigma S_t d\widehat{W}_t.$$

This means that

$$\begin{aligned} S(t) &= s_0 \exp\left\{\left(r - \delta - \frac{1}{2}\sigma^2\right) t + \sigma \widehat{W}_t\right\} \\ &= s_0 e^{-\delta t} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \widehat{W}_t\right\} \end{aligned}$$

This means that the price of an option on a stock which pays a continuous dividend yield can be computed using the Black-Scholes formula with today's stock price S_t, replaced with

$$S_t^{cy} = e^{-\delta(T-t)} S_t.$$

Remark 1 Assets that are usually considered to pay a continuous dividend yield are stock indices. It also turns out that exchange rates can be seen as assets paying a continuous dividend yield equal to r_f, where r_f is the foreign risk-free interest rate.

Adjusting the binomial tree to price options on stocks paying continuous dividend yield The expectation of S under Q when S pays a continuous dividend yield is

$$E^Q[S_t] = s_0 e^{(r-\delta)t}$$

To see this use that S_t/s_0 is lognormal with expectation $(r - \sigma^2/2)t$ and variance $\sigma^2 t$. This means that over a time step Δt in the binomial model the expected change in the asset price is

$$E^Q\left[S_{\Delta t}\right] = s_0 e^{(r-\delta)\Delta t}.$$

In the binomial model this is given by

$$E^Q\left[s_0 Z\right].$$

Thus the equation for q in this situation is

$$s_0 e^{(r-\delta)\Delta t} = q \cdot s_0 u + (1-q) \cdot s_0 d.$$

or

$$e^{(r-\delta)\Delta t} = q \cdot u + (1-q) \cdot d.$$

Remark 2 The corresponding equation for non-dividend paying stocks, or stocks paying discrete dividends can be written

$$e^{r\Delta t} = q \cdot u + (1-q) \cdot d.$$

For a concrete example of the calculations see Example 12.1 in "Fundamentals of Futures and Options Markets", or Figure 12.11 in "Options, Futures, and Other Derivatives".

Forward price of asset which provides a known yield We can now compute the forward price of $X = S_T$, when S pays a known dividend yield δ. From Proposition 1 of lecture 7 we know that

$$f(t;T,X) = \frac{\Pi_t[X]}{p(t,T)} = \frac{B_t E^Q[S_T/B_T|\mathcal{F}_t]}{p(t,T)}.$$

Using that $B_t = e^{rt}$ (which means that $p(t,T) = e^{-r(T-t)}$) we obtain that

$$f(t;T,X) = E^Q[S_T|\mathcal{F}_t] = e^{(r-\delta)(T-t)} S_t.$$

1.2 Futures options and the Black -76 formula

Suppose that we want to compute the price of a European call option with maturity T and strike price K written on a futures contract. The futures contract in turn is written on the underlying stock S and has delivery date T_1 such that $T < T_1$. The holder of such an option will at time T receive a long position in the futures contract and the amount

$$X = \max\{F(T;T_1) - K, 0\}.$$

Since the spot price of a futures contract is always zero we can forget about the long position in the futures contract, and concentrate on the T-claim X when pricing the option.
If we consider the standard Black-Scholes model we have that

$$F(t;T) = E^Q[S_T|\mathcal{F}_t] = B_T E^Q\left[\left.\frac{S_T}{B_T}\right|\mathcal{F}_t\right] = e^{r(T-t)} S_t.$$

and therefore

$$X = \max\{e^{r(T_1-T)} S_T - K, 0\} = e^{r(T_1-T)} \max\{S_T - e^{-r(T_1-T)} K, 0\}.$$

The option on the futures contract can thus be seen as $e^{r(T_1-T)}$ options on S with strike price $e^{-r(T_1-T)}K$ (and expiry date T). The Black-Scholes formula then yields

$$c_{fut} = e^{r(T_1-T)} \left\{ sN[d_1(t,s)] - e^{-r(T-t)} e^{-r(T_1-T)} KN[d_2] \right\}$$

or if we use that $s = e^{-r(T_1-t)} F(t;T_1)$ we obtain:

Proposition 1 (Black's formula) *The price, at t, of a European call option with strike price K and expiry date T on a futures contract (on the underlying asset S) with delivery date T_1 is given by*

$$c_{fut} = e^{-r(T-t)} \left(F(t;T_1) N[d_1] - KN[d_2] \right), \tag{1}$$

where
$$\begin{cases} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{F}{K}\right)+\frac{1}{2}\sigma^2(T-t)\right\}, \\ d_2 &= d_1-\sigma\sqrt{T-t}. \end{cases}$$
and N denotes the cumulative distribution function of the standard normal distribution, whereas $F = F(t;T_1)$.

For a concrete example of calculations using this formula, see Example 16.4 in "Fundamentals of Futures and Options Markets", or Example 17.6 in "Options, Futures, and Other Derivatives".

Remark 3 For the formula (1) to hold it is not really required that the spot price process S has the lognormal property, but rather that the futures price process does. This is useful when the underlying asset S is a commodity which is not ideally traded, e.g. electricity.

1.2.1 Using Black's model instead of Black-Scholes-Merton

Suppose that we want to price a European call option on the spot price S of an asset. If the asset is a consumption asset the Black-Scholes framework is in many cases not applicable (the underlying is not ideally traded and there may be other benefits than monetary that come from holding the asset). In this situation you can sometimes use the Black -76 formula although you are not trying to price a futures option. This is because at the delivery time T we have that the futures price satisfies

$$F(T;T,S_T) = S_T.$$

The payoff from a call option with maturity T and strike K on an underlying futures contract written on S with delivery time $T_1 \geq T$ is

$$X_{future} = \max\{F(T;T_1,S_{T_1}) - K, 0\}.$$

If the maturity of the option coincides with the delivery date, i.e. $T = T_1$, then

$$X_{future} = \max\{F(T;T,S_T) - K, 0\} = \max\{S_T - K, 0\} = X_{spot}$$

where X_{spot} is the payoff from a call option written on the spot price S. What is needed is therefore a futures contract written on the spot price with the same delivery date as the option has expiry date (both written on the same spot price of course).

Remark 4 Note that also the forward price f satisfies

$$f(T;T,S_T) = S_T.$$

If interest rates are assumed to be deterministic, also forward prices are Q-martingales and can be used in the Black -76 formula.

For some assets using futures prices rather than spot prices is a necessity, but it can save some work even for assets that can be handled within the Black-Scholes framework. For instance to price a European call option on an asset providing a continuous dividend yield of δ one should use the Black-Scholes formula with $S_0^{cy} = S_0e^{-\delta T}$ and will obtain

$$c_0 = S_0e^{-\delta T}N[d_1] - Ke^{-rT}N[d_2]$$

where

$$\begin{cases} d_1 &= \frac{1}{\sigma\sqrt{T}}\left\{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \sigma^2/2\right)T\right\}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{cases}$$

The forward price of such an asset is given by

$$f(0; T, S_T) = e^{(r-\delta)T} S_0.$$

If we insert $S_0 = e^{-(r-\delta)T} f_0$ into the above formula we obtain

$$c_0 = e^{-rT}\left(f_0 N[d_1] - K N[d_2]\right).$$

where

$$\begin{cases} d_1 &= \frac{1}{\sigma\sqrt{T}}\left\{\ln\left(\frac{f_0}{K}\right) + \frac{1}{2}\sigma^2 T\right\}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{cases}$$

which is the Black -76 formula. We see that with this formula there is no need to estimate δ! In general using the Black -76 formula makes it unnecessary to estimate dividends, storage costs, and yield for the underlying asset.

1.2.2 Using a binomial tree for options on futures contracts

Recall that the futures price process is a martingale under Q (see Proposition 3 of lecture 7). This means that when we build the tree we should make sure that

$$f = E^Q[F_{t+1} | F_t = f]$$

or, more explicitly

$$f = q \cdot fu + (1 - q) \cdot fd.$$

For a concrete example of the computations, see Example 12.3 in “Fundamentals of Futures and Options Markets” and Figure 12.13 in “Options, Futures and Other Derivatives”.

1.3 Hedging and the Greeks

You may have heard that options are risky, well are they? Let us consider a European call option on an underlying stock.

Example 1 Let us consider an extreme example. Consider a European call option on an underlying stock. Say that today's stock price is \$100. Assume that the call option has a strike price of K which is \$99 and that and should be exercised tomorrow. This means that the option price today is roughly \$1.

Assume now that the stock price goes down \$1 until tomorrow. The option then becomes worthless, i.e. the option price goes down \$1.

This means that a drop of 1% in the stock price results in a drop of almost 100% in the option price! In this sense option are risky. □

Let us look at another example.

Example 2 Assume that we have sold a European call option on 100 000 ABC stocks for \$2 million.

Suppose that we have the following data for the market

$$S = 365, \quad K = 370, \quad \sigma = 0.2, \quad r = 7\%, \quad T - t = 1/4.$$

Then the option price is given by

$$\begin{aligned} c(t, S) &= 100\,000 \times \\ &\times \{SN[d_1(t, S)] - Ke^{-r(T-t)}N[d_2(t, S)]\} \\ &= 100\,000 \times 15.264 \end{aligned}$$

We thus seem to have made a profit of

$$2\,000\,000 - 1\,526\,400 = 473\,600 \text{ dollars},$$

but . . . we are exposed to financial risk!

To deal with the financial risk we can have a number of strategies.

The first of which is called taking a *naked position*, i.e. do nothing! There are then different scenarios at time $t = T$

- If $S_T < 370$ the option will not be exercised.
  Our net earnings: \$2 000 000.
- If $S_T > 370$ the option will be exercised.
  We will have to buy 100 000 stocks at the price S_T and sell them at the price \$370
  Our net earnings are then: $\$2\,000\,000 - 100\,000 \times (S_T - 370)$
  - If S_T=385: \$500 000
  - If S_T=395: \$−500 000

The second strategy is called taking a *covered position*. This means that we buy 100000 stocks today. Again there are different scenarios at time $t = T$.

- If $S_T < 370$ the option will not be exercised.
  Our net earnings: $\$2\,000\,000 + 100\,000 \times (S_T - 365)$.
  - If S_T=360: \$1 500 000
  - If S_T=340: \$−500 000
- If $S_T > 370$ the option will be exercised.
  Our net earnings: $\$2\,000\,000 + 100\,000 \times (370 - 365) = \$2\,500\,000$.

To sum up: If we take a naked position it will always result in a profit if $S_T < K$, but can result in a loss if $S_T > K$. If we take a covered position it will always result in a profit if $S_T > K$, but can result in a loss if $S_T < K$. Thus neither strategy gives complete protection against financial risk, which is of course also hoping for a bit too much. □

The purpose of hedging is to *limit* or at least to *lower* the financial risk. Depending on your attitude to risk and the information available to you, you can choose to set up

- a perfect hedge
- a delta hedge
- a gamma hedge or
- use some other strategy ...

Fix a T-claim X.

Definition 1 *A self-financing portfolio h is a* perfect hedge *against X if*

$$V_T^S(h) = X \quad \textit{with probability 1 under } P.$$

If you set up a perfect hedge you know exactly how much money you will make. The problem with setting up a perfect hedge is that in a continuous time framework the replicating portfolio requires rebalancing in continuous time, which leads to high transactions costs! So what can we do instead?

1.3.1 The Greeks

Let $P(t, s)$ denote the pricing function of a portfolio based on **one** underlying asset with price process S_t.

Definition 2 *The Greeks are defined in the following way:*

$$\begin{aligned} \Delta &= \frac{\partial P}{\partial s} & \theta &= \frac{\partial P}{\partial t} \\ \Gamma &= \frac{\partial^2 P}{\partial s^2} & \rho &= \frac{\partial P}{\partial r} \\ & & \mathcal{V} &= \frac{\partial P}{\partial \sigma} \end{aligned}$$

The letter $\mathcal{V}$ is not really a Greek letter at all, but it is called "vega". The "Greeks" are sensitivity measures. Δ and Γ reflect the portfolio's sensitivity to small changes in the price of the underlying (financial risk), whereas ρ and $\mathcal{V}$ reflect the portfolio's sensitivity to misspecifications of the model (recall that r and σ should be constant in the Black-Scholes framework)!

1.3.2 Delta hedging

The goal of delta hedging is to make the portfolio insensitive to small changes in the price of the underlying.

Definition 3 *A portfolio with $\Delta = 0$ is said to be* delta neutral.

The idea now is to add a derivative to the original portfolio. Since the price of the derivative is perfectly correlated with the price of the underlying it should be possible to choose portfolio weights so as to make the modified portfolio delta neutral.

Definition 4 *A position in the derivative is a* delta hedge *for the portfolio if the modified portfolio (original portfolio + derivative) is delta neutral.*

We will use the following notation:

$P(t,s)$ the pricing function of the original portfolio

$F(t,s)$ the pricing function of the derivative

y the number of derivatives to add to the portfolio.

The value process $V(t,s)$ of the modified portfolio is

$$V(t,s) = P(t,s) + y \cdot F(t,s).$$

The modified portfolio is delta neutral if

$$\frac{\partial V}{\partial s} = 0$$

i.e.

$$\frac{\partial P}{\partial s} + y \frac{\partial F}{\partial s} = 0.$$

We can now solve for y! The solution is

$$y = -\frac{\Delta_P}{\Delta_F}$$

Example 3 Suppose that we have sold a derivative with price function $G(t,s)$ and that we wish to hedge it using the underlying itself. Then we have that

$$P(t,s) = -G(t,s)$$

and that

$$F(t,s) = s.$$

For y we obtain

$$y = \Delta_G.$$

The delta of the derivative gives us the number of underlying assets we have to buy to hedge the derivative! □

There is a problem with the delta hedging strategy: the portfolio has to be rebalanced when the price of the underlying changes, because as the price changes Δ will change. One can actually show the following.

Proposition 2 *For a continuously rebalanced delta hedge in the underlying, the value of the underlying and of the risk-less asset (used to keep things self-financing) will replicate the derivative.*

This means that the continuously rebalanced delta hedge described above is a perfect hedge!

Proposition 3 *In the Black-Scholes framework the delta of a European call option written on a non-dividend paying stock is*

$$\Delta_c = N(d_1),$$

where N denotes the cumulative distribution function of the standard normal distribution, and d_1 is given by

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}.$$

Proof: Just take derivatives! Note that it is not quite as easy as it seems, since d_1 and d_2 in Proposition 2 in lecture 8 also depend on s. □

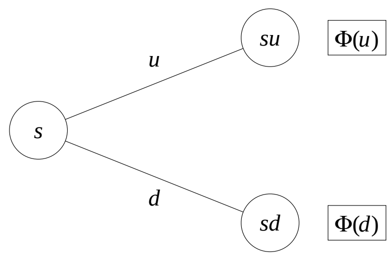
Estimating Delta using a binomial tree To compute Δ in a binomial model recall that

$$\Delta = \frac{\partial P}{\partial s}.$$

where P denotes the price of the original portfolio and for the binomial model we will think of this as consisting of a single derivative. This means that

$$\Delta \approx \frac{\Delta P}{\Delta s}.$$

Now consider the following one step situation in a binomial tree.



Here $\phi(u)$ is the price of the derivative if the stock price goes up, and $\phi(d)$ is the price of the derivative if the stock price goes down. We then have that

$$\Delta = \frac{\Delta P}{\Delta s} = \frac{\phi(u) - \phi(d)}{su - sd} = \frac{1}{s}\frac{\phi(u) - \phi(d)}{u - d}.$$

For concrete computations, see Section 12.6 in “Fundamentals of Futures and Options Markets”, or “Options, Futures, and Other Derivatives”.

Remark 5 The letter y in the computations above was chosen for mnemonic reasons. Recall that in the replicating portfolio we used x for the amount of SEK in the bank account and y for the number of assets in the replicating portfolio. If you compare the expression for y in the replicating portfolio given in the proof of Proposition 2 in lecture 1, and the expression for Δ found above you will see that they are exactly the same!

1.3.3 Gamma hedging

A delta hedge is rebalanced because S and with that also Δ is changed. Γ is a measure of how sensitive Δ is to changes in S

$$\Gamma = \frac{\partial^2 P}{\partial s^2} = \frac{\partial \Delta}{\partial s}.$$

The goal now is to make the portfolio both delta and gamma neutral. The idea is to add **two** derivatives to the portfolio to be able to achieve this.
We will use the following notation:

$P(t, s)$ pricing function of the portfolio

$F(t, s)$ pricing function of derivative 1

$G(t, s)$ pricing function of derivative 2

y_F the number of derivatives of type 1 to add to the portfolio

y_G the number of derivatives of type 2 to add to the portfolio

The value process $V(t,s)$ of the modified portfolio is

$$V(t,s) = P(t,s) + y_F \cdot F(t,s) + y_G \cdot G(t,s).$$

We want the following to hold

$$\frac{\partial V}{\partial s} = 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial s^2} = 0.$$

This yields the following system of equations

$$\begin{cases} \Delta_P + y_F \Delta_F + y_G \Delta_G &= 0, \\ \Gamma_P + y_F \Gamma_F + y_G \Gamma_G &= 0. \end{cases}$$

Solve for y_F and y_G!
Note that for the underlying itself we have

$$\Delta_S = 1 \qquad \text{and} \qquad \Gamma_S = 0$$

If you choose $G(t,s) = s$ you will get a triangular system which is easy to solve

$$\begin{cases} y_F &= -\dfrac{\Gamma_P}{\Gamma_F}, \\ y_S &= \dfrac{\Delta_F \Gamma_P}{\Gamma_F} - \Delta_P. \end{cases}$$