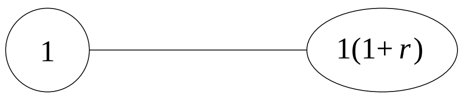
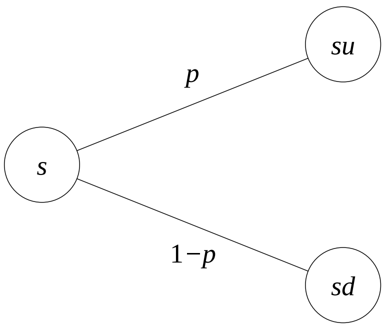
A one period model:

Time structure: $t = 0, 1$

Bond:



Stock:



Portfolio: $h = (x, y)$

Value process: $V^h(t) = xB(t) + yS(t)$

Contingent claim: stochastic variable

$$X = \phi(S(1)) \text{ or } X = \phi(Z)$$

Hedge/replicating portfolio for X: h such that

$$V^h(1) = X, \text{ with probability 1}$$

Arbitrage portfolio: h such that

$$\begin{aligned} V^h(0) &= 0, \\ P\left(V^h(1) \geq 0\right) &= 1, \\ P\left(V^h(1) > 0\right) &> 0, \end{aligned}$$

Prop.: If X is replicated by h then all prices except for

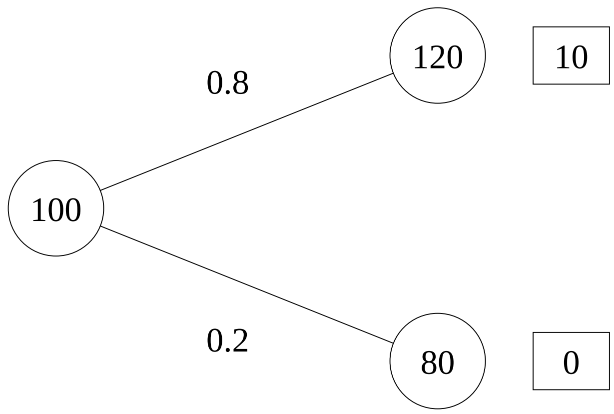
$$\Pi(t; X) = V^h(t)$$

give rise to arbitrage opportunities.

Example 1:

Parameters $r = 0$, $u = 1.2$, $d = 0.8$, $p = 0.8$.

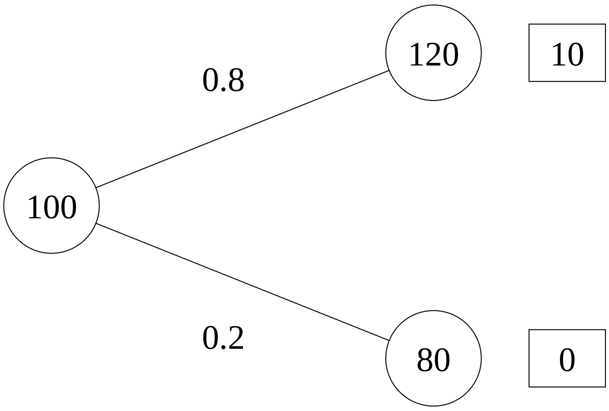
We want to price a European call option with strike price $K = 110$.



Two common guesses for the price of the option:

Guess 1: $\Pi(1) = \frac{1}{1+0}\{0.8 \cdot 10 + 0.2 \cdot 0 = 8\}$.

Guess 2: There is no correct price.



We have that for $h = \left(-20, \dfrac{1}{4}\right)$

$$
\begin{aligned}
V^h(1) &= x \cdot B(1) + y \cdot S(1) \\
&= \begin{cases} -20 \cdot 1 + \frac{1}{4} \cdot 120 = 10 & \text{if } S(1) = 120 \\ -20 \cdot 1 + \frac{1}{4} \cdot 80 = 0 & \text{if } S(1) = 80 \end{cases}
\end{aligned}
$$

thus h replicates the payoff of the option.

Furthermore

$$
\begin{aligned}
V^h(0) &= x \cdot B(0) + y \cdot S(0) \\
&= -20 \cdot 1 + \tfrac{1}{4} \cdot 100 = 5.
\end{aligned}
$$

Thus a fair price for the option is 5.

Prop.: The model is **complete** (all claims are reachable) if $u > d$.

Proof: Solve

$$\begin{aligned}(1+r)x + suy &= \phi(u)\\ (1+r)x + sdy &= \phi(d)\end{aligned}$$

This yields

$$\begin{aligned}x &= \frac{1}{1+r}\cdot\frac{u\phi(d) - d\phi(u)}{u-d}\\ y &= \frac{1}{s}\cdot\frac{\phi(u)-\phi(d)}{u-d}\end{aligned}$$

Summary:

- We compute the price **as if** we live in a risk neutral world.

  Note: We use martingale probabilities!

- This does **not** mean that we live in a risk neutral world (or that we think that we do).

- The valuation formula is valid regardless of your attitude towards risk, as long as you prefer more risk less profit to less.

  The valuation formula is therefore often referred to as preference free.