KTH Mathematics

Examination in SF2701 Financial Mathematics, May 29, 2012, 14:00–19:00.

Examiner: Filip Lindskog, tel. 790 7217, e-mail: lindskog@kth.se

Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. Only an answer, without an explanation, will give 0 points. Interest rates refer to continuous compounding. log denotes the natural logarithm so that $\log e = 1$. Unless stated otherwise, you may take long and short positions corresponding to fractions of units of assets.

Good luck!

- - - - - - - - - - - - - - - - - - - -

Problem 1

Consider three interest rate swaps with the common nominal amount $L = 1,000,000$ dollars and maturities $1, 2, 3$ years. Each swap pays yearly fixed-rate payments and floating-rate payments every six months. The first floating-rate payment is in six months. The k-year swap pays the fixed amount $c_k L$ yearly: the first payment is in one year and the last payment is in k years. The c_ks are given by $c_1 = 0.0054$, $c_2 = 0.0066$, $c_3 = 0.0087$.

(a) Determine the 1-year, 2-year, and 3-year zero rates from the swap rates. (5 p)

(b) Determine the current forward price for delivery in just after 12 months of the remaining cash flow of the 3-year swap for the fixed-rate payer/floating-rate receiver. (5 p)

Problem 2

Consider a share of a stock with costs 100 dollars today, and which gives a known dividend amount 4 dollars in 3 months. There are no other dividend payments. Suppose that there is a three-month zero-coupon bond with zero rate 3% per year and a six-month zero-coupon bond with zero rate 4% per year. Suppose further that you have taken a short position in a forward contract for delivery of one share in 6 months and that the forward price is 99 dollars per share.

Determine a position with zero net cash flow at all times prior to six months from today in the stock and in zero-coupon bonds that makes the net cash flow, including that of the forward contract, in six months strictly positive with certainty. Compute the cash flow of this position, including the forward contract, in six months from today. (10 p)

Problem 3

Consider a coffee buyer who is planning to buy a yet unknown number of pounds of coffee beans in six months at the prevailing spot price, in cents per pound, at that time. There are futures contracts available for delivery of 37,500 pounds of coffee beans in six months. The current futures price is 180 cents per pound.

Assume that the interest rates are zero and that the coffee buyer has sufficient cash to handle margin calls. Assume further that the coffee buyer believes that the expected

value and standard deviation of number of pounds of coffee beans to buy is $100,000$ and $10,000$, respectively, and that the expected value and standard deviation of the spot price in cents per pound in six months is 180 and 20, respectively. Assume further that quantity to buy and the spot price in six months are independent.
Denote by the effective price the price paid for the coffee beans including the gain/loss from the futures position. The futures position is chosen so that the variance of the effective price is minimized. Determine the futures position (long/short and size). (10 p)

Problem 4

The current three-month zero rate is 4% per year and the current six-month zero rate is 5% per year. The three-month zero rate in three months is either 3% or 6% per year. Determine the current price of a European call option maturing in three months with strike price 990 dollars on the value in three months of a zero-coupon bond maturing in six months with face value 1000 dollars. (10 p)

Problem 5

Consider three American call options on the share price of a stock in three months. The current call option prices corresponding to the strike prices $200, 205, 210$ dollars are $7, 6, 4$ dollars, respectively. Determine a strategy that produces a risk-free profit from trading in the American call options. (10 p)

Problem 6

Use the Ho-Lee binomial short interest rate model,

$$
\begin{aligned}
r_1 &= -\log(Z_1), \\
r_k &= \log(Z_{k-1}/Z_k) + \log(\cosh((k-1)\sigma)) + \sigma \sum_{j=2}^{k} b_j, \quad k \geq 2,
\end{aligned}
$$

where the b_js are independent and takes the values ± 1, each with probability $1/2$, under the futures probability distribution, to compute the price at time 0 of a European put option maturing at time 2 with strike price 90 dollars on a zero-coupon bond maturing at time 4 with face value 100 dollars. The zero rates for maturity at times $1, 2, 3, 4$ are $0.03, 0.035, 0.04, 0.045$, respectively, per time-step. The parameter σ is 0.01. (10 p)

Problem 1

The discount factors are

$$\begin{aligned} Z_1 &= \frac{1}{1+c_1} = 0.9946290, \\ Z_2 &= \frac{1-c_2Z_1}{1+c_2} = 0.9869218, \\ Z_3 &= \frac{1-c_3Z_1-c_3Z_2}{1+c_3} = 0.9742842. \end{aligned}$$

The corresponding zero rates $-\log(Z_k)/k$ are $0.005385472, 0.006582254, 0.008684065$. The present value of the 3-year swap's fixed payments after 12 months from today is $(Z_2c_3 + Z_3c_3)L$. The present value of the 3-year swap's floating payments after 12 months from today is $(Z_1 - Z_3)L$. The forward price is therefore

$$Z_1^{-1}(Z_1 - Z_3 - Z_2c_3 - Z_3c_3)L = 3300.$$

Problem 2

At current time, buy on share of the stock, short sell three-months zero-coupon bonds worth $de^{-r_{1/4}/4}$ today, and short sell six-months zero-coupon bonds worth $S_0 - de^{-r_{1/4}/4}$ today. The position in the three-months zero-coupon bond cancels the dividend payment. The cash flow in six months is

$$G_0 - (S_0 - de^{-r_{1/4}/4})e^{r_{1/2}/2} = 99 - (100 - 4e^{-0.03/4})e^{0.04/2} = 1.03018.$$

Problem 3

Let X and S be the quantity and spot price, respectively. Let $n = 37,500$ and let h the the size, including sign, of the futures position. Write $Y = XS - nh(S - F_0)$ for the effective price. Then

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}[S^2]\,\mathrm{E}[(X-nh)^2] - \mathrm{E}[S]^2\,\mathrm{E}[X-nh]^2 \\ &= (\mathrm{Var}(S) + \mathrm{E}[S]^2)(\mathrm{Var}(X) + (\mathrm{E}[X]-nh)^2) + \mathrm{E}[S^2](\mathrm{E}[X]-nh)^2 \\ &= (n^2h^2 - 2n\,\mathrm{E}[X]h + \mathrm{E}[X]^2)(\mathrm{Var}(S) + 2\,\mathrm{E}[S]^2) + \mathrm{Var}(X)(\mathrm{Var}(S) + \mathrm{E}[S]^2) \end{aligned}$$

and considering $\mathrm{Var}(Y)$ as a function of h, computing it's derivative, setting the expression to zero, and solving it for h gives $h = \mathrm{E}[X]/n = 8/3$ (long position). Therefore the expected value and variance of the effective price are

$$\begin{aligned} &\mathrm{E}[X]\,\mathrm{E}[S] - \mathrm{E}[X](\mathrm{E}[S] - F_0) = F_0\,\mathrm{E}[X] = 180,000 \text{ dollars} \\ &\sqrt{\mathrm{Var}(X)(\mathrm{Var}(S) + \mathrm{E}[S]^2)} = 10,000 \times \sqrt{400 + 180^2} = 18,110.77 \text{ dollars.} \end{aligned}$$

Problem 4

According to the problem statement:

$$e^{-0.05/2} = \widehat{\mathrm{E}}[e^{-(r_1+r_2)/4}] = e^{-0.04/4}\Big(qe^{-0.03/4} + (1-q)e^{-0.06/4}\Big)$$

which gives $q = 0$. Since $Z_{1/4,1/2} = \widehat{\mathrm{E}}_{1/4}[e^{-r_2/4}] = e^{-r_2/4}$ we have

$$\begin{aligned}
P_0^{(1/4)}[\max(1000Z_{1/4,1/2} - K, 0)] &= \widehat{\mathrm{E}}[\max(1000Z_{1/4,1/2} - 990, 0) \cdot e^{-r_1/4}] \\
&= q\Big(1000e^{-0.03/4} - 990\Big)e^{-0.04/4} \\
&\quad + (1-q) \cdot 0 \cdot e^{-0.04/4} \\
&= 0.
\end{aligned}$$

Problem 5

At time 0 buy 1 call options with strike price 200, buy 1 call options with strike price 210, and short-sell 2 call options with strike price 205. This gives a strictly positive cash flow $-7 + 12 - 4 = 1$ at time 0. Whenever the short-sold call option is exercised, exercise the other two call options. This produces the cash flow

$$C = \max(S - 200, 0) + \max(S - 210, 0) - \max((S - 200)/2 + (S - 210)/2, 0)$$

at the exercise time of the short-sold call option and cancels the call option position, where S denotes the share price at the exercise time. Since $\max(x, 0)$ is a convex function,

$$\max((S - 200)/2 + (S - 210)/2, 0) \leq \frac{1}{2}\max(S - 200, 0) + \frac{1}{2}\max(S - 210, 0).$$

Therefore $C \geq 0$ so the strategy produces a strictly positive cash flow at time 0, a nonnegative cash flow at the exercise time of the short-sold call option, and no other cash flows.

Problem 6

The tree of short rates:

```
0.03 0.05005 0.07019999 0.09044993
     0.03005 0.05019999 0.07044993
             0.03019999 0.05044993
                        0.03044993
```

The tree of bond prices:

```
83.52702 83.48945 86.01918 91.35201
         88.65215 89.52969 93.19744
                  93.18346 95.08015
                           97.00090
```

The tree of option prices:

```
1.137903 2.1169198 3.9808225
         0.2281953 0.4703132
                   0.0000000
```