KTH Mathematics

Examination in SF2701 Financial Mathematics, August 13, 2012, 8:00–13:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. Only an answer, without an explanation, will give 0 points. Interest rates refer to continuous compounding. log denotes the natural logarithm so that $\log e = 1$. Unless stated otherwise, you may take long and short positions corresponding to fractions of units of assets.

Good luck!

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Problem 1

Consider three interest rate swaps with the common nominal amount $L = 1,000,000$ dollars and maturities $1, 2, 3$ years. Each swap pays yearly fixed-rate payments and floating-rate payments every six months. The first floating-rate payment is in six months. The k-year swap pays the fixed amount $c_k L$ yearly: the first payment is in one year and the last payment is in k years. The c_ks are given by $c_1 = 0.0054$, $c_2 = 0.0066$, $c_3 = 0.0087$.

Suppose that, in just after 12 months from now, the 1-year, 2-year, and 3-year swap rates are given by $c'_1 = 0.0060$, $c'_2 = 0.0070$, $c'_3 = 0.0090$. Determine the value at that time of the original 3-year swap contract for the fixed-rate payer/floating-rate receiver. (10 p)

Problem 2

Consider a binary call option on a dividend paying stock which pays $1,000$ dollars if the share price in four months exceeds 100 dollars. The current price of the binary call option is 272 dollars, the current four-month zero rate is 3% per year, the current share price is 95 dollars, and the share pays a dividend of 3 dollars in two months.

Determine the current price of a binary put option which pays 500 dollars if the share price in four months does not exceed 100 dollars. (10 p)

Problem 3

Consider a coffee producer who is planning to sell a yet unknown number X pounds of coffee beans in six months at the prevailing spot price S, in cents per pound, at that time. There are futures contracts available for delivery of n pounds of coffee beans in six months. The current futures price is F_0 cents per pound. Assume that the interest rates are zero and that the coffee buyer has sufficient cash to handle margin calls. Assume further that the coffee producer believes that the expected value and variance of the number of pounds of coffee beans to sell is $\mathrm{E}[X]$ and $\mathrm{Var}(X)$, respectively, and that the expected value and variance of the spot price, in cents per pound, in six months is $\mathrm{E}[S]$ and $\mathrm{Var}(S)$, respectively. Assume further that quantity to sell and the spot price in six months are independent.

Determine the futures position (long/short and size) that minimizes the variance of the net income (including the gain/loss from the futures position) in six months, and determine the minimal variance of the net income. (10 p)

Problem 4

The current three-month zero rate is 4% per year and the current six-month zero rate is 5% per year. The three-month zero rate in three months is either 3%, 6% or 9% per year. The current price of a European call option maturing in three months with strike price 990 dollars on the value in three months of a zero-coupon bond maturing in six months with face value $1,000$ dollars is 0.75 dollars.

Determine the current price of a European put option maturing in three months with strike price 990 dollars on the value in three months of a zero-coupon bond maturing in six months with face value $1,000$ dollars. (10 p)

Problem 5

The current 18-month interest rate of the US dollar is 2% per year and the current 18-month interest rate of the euro is 4% per year. The current exchange rate of the US dollar and the euro is 0.8, meaning that one US dollar can be exchanged for 0.8 euros. An investor has a long position in a forward contract for delivery of $1,000,000$ US dollars in 18 months. The forward price is $830,000$ euros.

Determine how the investor, or his counterpart in the forward contract, can make a risk-free profit by combining the cash flow of the forward contract with an appropriate trading strategy in currency and zero-coupon bonds. (10 p)

Problem 6

Use the Ho-Lee binomial short interest rate model,

$$r_1 = -\log(Z_1),$$

$$r_k = \log(Z_{k-1}/Z_k) + \log(\cosh((k-1)\sigma)) + \sigma \sum_{j=2}^{k} b_j, \quad k \geq 2,$$

where the b_js are independent and takes the values ± 1, each with probability $1/2$, under the futures probability distribution, to compute the price at time 0 of an American call option maturing at time 2 with strike price 91 dollars on a zero-coupon bond maturing at time 4 with face value 100 dollars. The zero rates for maturity at times $1, 2, 3, 4$ are $0.045, 0.04, 0.035, 0.035$, respectively, per time-step. The parameter σ is 0.01. (10 p)

Problem 1

Given the assumed future swap rates in 12 months, the discount factors in 12 months are

$$\begin{aligned} Z_1' &= \frac{1}{1+c_1'} = 0.9940358, \\ Z_2' &= \frac{1-c_2'Z_1}{1+c_2'} = 0.9861388, \\ Z_3' &= \frac{1-c_3'Z_1-c_3'Z_2}{1+c_3'} = 0.9734177. \end{aligned}$$

The present values P_{fi} and P_{fl} at that time for the remaining fixed and floating rate payments of the 3-year swap then takes the values

$$\mathrm{P}_{fi} = (Z_1'+Z_2')c_3L = 17,227.52 \quad \text{and} \quad \mathrm{P}_{fl} = (1-Z_2')L = 13,861.22.$$

The present value of the swap for the fixed-rate payer/floating-rate receiver is therefore $\mathrm{P}_{fl} - \mathrm{P}_{fi} = -3,366.3$ dollars.

Problem 2

Since $1 = I\{S_{1/3} > 100\} + I\{S_{1/3} \leq 100\}$ it holds that $e^{-0.03/3} = c + p$, where $c = 0.272$ is the price of a binary call option that pays one dollar if $S_{1/3} > 100$ and p is the price of a binary put option that pays one dollar if $S_{1/3} \leq 100$. The sought put option price is therefore $500p = 500(e^{-0.03/3} - c) = 359.0249$ dollars.

Problem 3

Let h the the size, including sign, of the futures position. Write $Y = XS + nh(S - F_0)$ for the net income. Then

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}[S^2]\,\mathrm{E}[(X+nh)^2] - \mathrm{E}[S]^2\,\mathrm{E}[X+nh]^2 \\ &= (\mathrm{Var}(S) + \mathrm{E}[S]^2)(\mathrm{Var}(X) + (\mathrm{E}[X]+nh)^2) - \mathrm{E}[S^2](\mathrm{E}[X]+nh)^2 \\ &= \mathrm{Var}(S)(\mathrm{Var}(X) + (\mathrm{E}[X]+nh)^2) + \mathrm{E}[S]^2\,\mathrm{Var}(X) \end{aligned}$$

which is minimized for $h = -\mathrm{E}[X]/n$ (short position of size $\mathrm{E}[X]/n$). The variance of the net income with the optimal futures position is therefore $\mathrm{Var}(X)(\mathrm{Var}(S) + \mathrm{E}[S]^2)$.

Problem 4

According to the problem statement:

$$e^{-0.05/2} = \widehat{\mathrm{E}}[e^{-(r_1+r_2)/4}] = e^{-0.04/4}\Big(p_1 e^{-0.03/4} + p_2 e^{-0.06/4} + (1-p_1-p_2)e^{-0.09/4}\Big).$$

Since $Z_{1/4,1/2} = \widehat{\mathrm{E}}_{1/4}[e^{-r_2/4}] = e^{-r_2/4}$ we have

$$\begin{aligned} 0.75 &= P_0^{(1/4)}[\max(1000Z_{1/4,1/2} - 990, 0)] \\ &= \widehat{\mathrm{E}}[\max(1000Z_{1/4,1/2} - 990, 0)\cdot e^{-r_1/4}] \\ &= p_1\Big(1000e^{-0.03/4} - 990\Big)e^{-0.04/4} \\ &\quad + p_2 \cdot 0 \cdot e^{-0.04/4} + (1-p_1-p_2)\cdot 0 \cdot e^{-0.04/4} \\ &= p_1\Big(1000e^{-0.03/4} - 990\Big)e^{-0.04/4}. \end{aligned}$$

Solving for p_1 gives $p_1 = 0.2996524$. Therefore,

$$p_2 = \frac{e^{-0.03/2} - p_1(e^{-0.03/4} - e^{-0.09/4}) - e^{-0.09/4}}{e^{-0.06/4} - e^{-0.09/4}} = 0.3984394$$

which yields

$$\begin{aligned}
&P_0^{(1/4)}[\max(990 - 1000Z_{1/4,1/2}, 0)] \\
&\quad = \widehat{\mathrm{E}}[\max(990 - 1000Z_{1/4,1/2}, 0) \cdot e^{-r_1/4}] \\
&\quad = p_1 \cdot 0 \cdot e^{-0.04/4} \\
&\qquad + p_2\Big(990 - 1000e^{-0.06/4}\Big)e^{-0.04/4} \\
&\qquad + (1 - p_1 - p_2)\Big(990 - 1000e^{-0.09/4}\Big)e^{-0.04/4} \\
&\quad = 5.589423.
\end{aligned}$$

Problem 5

Following the argument in Example 5, page 10, in the lecture notes we find that the forward price consistent with the current exchange rate and interest rates should be $0.8e^{0.03} < 0.83$ euro per dollar. Therefore the investor's counterpart in the forward contract can make a risk-free profit by taking a long position in a synthetic forward contract. Let $n = 10^6$, $r = 0.04$, $\rho = 0.02$, $t = 3/2$, $X_0 = 0.8$. (a): short-sell euro-zero-coupon bonds maturing in 18 months worth $nX_0e^{-\rho t}$ today. (b): use the euro income to buy dollars today, buy dollar-zero-coupon bonds for these dollars, and exchange the bond payoff in 18 months to euros. (c): the cash flow of the short forward position.

	0	t
(a)	$nX_0e^{-\rho t}$	$-nX_0e^{(r-\rho)t}$
(b)	$-nX_0e^{-\rho t}$	nX_t
(c)	0	$nG - nX_t$
net	0	$n(G - X_0e^{(r-\rho)t}) > 0$

The net cash flow is 0 today and $n(G - X_0e^{(r-\rho)t}) = 5636.37$ euros in 18 months.

Problem 6

The tree of short rates:

0	1	2	3
0.045	0.04505	0.045199987	0.065449933
	0.02505	0.025199987	0.045449933
		0.005199987	0.025449933
			0.005449933

The tree of bond prices:

0	1	2	3
86.93582	88.20999	90.42948	93.66459
	93.66459	94.11997	95.55674
		97.96108	97.48712
			99.45649

Since the bond price is increasing in time early exercise will never be profitable. Tree of option prices:

0	1	2
3.062585	1.491268	0
	4.915830	3.119972
		6.961081