KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 20 2014 14.00-19.00.

Answers and brief solutions.

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1. (a) This exercise can be solved in two ways.

i. Risk-neutral valuation. The martingale measure should satisfy

$$e^{r\Delta t} S_t = q S^u_{t+\Delta t} + (1-q) S^d_{t+\Delta t},$$

so

$$q = \frac{e^{r\Delta t} S_t - S^d_{t+\Delta t}}{S^u_{t+\Delta t} - S^d_{t+\Delta t}}.$$

With numbers

$$q = \frac{e^{0.05 \cdot 0.25} 50 - 47}{54 - 47} \approx 0.5184175.$$

The price of the option in three months is

$$\Pi(T) = X = \max\{K - S_T\} = \max\{51 - S_T\} = \begin{cases} 0 & \text{if } S_T = 54 \\ 4 & \text{if } S_T = 47 \end{cases}$$

The price at time $t = 0$ is then found as

$$\Pi(0) = E^Q\left[\frac{\Pi(T)}{B_T}\right] = e^{-rT} E^Q[X]$$

or with numbers

$$\Pi(0) = e^{-0.05 \cdot 0.25}\{0.5184175 \cdot 0 + (1 - 0.5184175) \cdot 4\} \approx 1.90.$$

ii. Replicating portfolio. The number of stocks in the replicating portfolio is $y = \Delta$, i.e.

$$y = \Delta = \frac{\Delta\Pi}{\Delta S} = \frac{0-4}{54-47} = \frac{-4}{7}.$$

To find the amount of cash x you should have in the bank account solve

$$x e^{r\Delta t} + y S^u_{t+\Delta t} = \Pi^u(t + \Delta t)$$

with numbers

$$x e^{0.05 \cdot 0.25} + \frac{-4}{7} \cdot 54 = 0.$$

This yields $x = e^{-0.05 \cdot 0.25} 216/7$ and the value of the option at time $t = 0$ is equal to the value of the replicating portfolio at time $t = 0$, that is

$$x + y S_0 = e^{-0.05 \cdot 0.25} 216/7 - \frac{4}{7} \cdot 50 \approx 1.90.$$

(b) If we denote by $C(t, S_t)$ the price at time t of a European call option with strike price K and expiry date T written on the stock with price S_t at time t, and by $P(t, S_t)$ the price at time t of a European put option with the same strike price and expiry date as the call, and also having the stock as underlying, then according to put-call parity we have

$$P(t, S_t) = Ke^{-r(T-t)} + C(t, S_t) - S_t \text{ or } C(t, S_t) = P(t, S_t) - Ke^{-r(T-t)} + S_t$$

Using this we obtain that the price of the call option is

$$C(0, S_0) = P(0, S_0) - Ke^{-rT} + S_0,$$

where $S_0 = 78$, $P(0, S_0) = 6.5$, $K = 80$, $r = 0.03$ and $T = 0.5$. Inserting the numbers we find that the value of the call option is

$$C(0, S_0) = 6.5 - 80e^{-0.03 \cdot 0.5} + 78 \approx 5.6910,$$

i.e. \$5.69.

2. (a) We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/2 = 1/4 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.1331 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.8825 \end{aligned}$$

and the tree for the stock price is therefore

		128.4025
	113.3148	
100.0000		100
	88.2497	
		77.8801

Now the option price tree can be computed using

$$q = \frac{e^{(r-\delta)\Delta t} - d}{u - d} \approx 0.4489,$$

note that the dividend yield comes in to play when computing the martingale probabilities, and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0075}$$

and the result is

		28.4025
	13.3148	
5.9323		0.0000
	0.0000	
		0.0000

In each node, that is not an end node, the value is obtained as

$$\max\{S_t - 100, \frac{1}{1.0075}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_t is the current stock price, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 13.3148. The price of the option is thus 5.9323.

(b) The price of a call option will increase if there are no dividends, since the option is more likely to be in the money if the stock pays no dividends (the stock price will be higher).

3. (a) i. We have an underlying asset paying a continuous dividend yield of $\delta = 0.04$. Use the Black-Scholes formula with parameters

$$s = S_0 e^{-\delta T} = 100e^{-0.04 \cdot 0.75}, \quad K = 99, \quad \sigma = 0.2, \quad r = 0.03, \quad T = 0.75$$

to obtain

$$c_{yield} = 6.8131.$$

ii. The price of the American call option can not be easily determined. In order to be able to price an American call option one needs to be in the standard Black-Scholes framework, where the underlying asset pays no dividends and the interest rate is positive. Then price of the American call option is equal to the price of the corresponding European call option.

(b) Use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)}F_t$) with parameters

$$F_0 = 100, \quad K = 103, \quad \sigma = 0.165, \quad r = 0.05, \quad T = 0.5$$

or the Black-Scholes formula with parameters

$$s = F_0 e^{-rT} = 100e^{-0.05 \cdot 0.5}, \quad K = 103, \quad \sigma = 0.165, \quad r = 0.05, \quad T = 0.5.$$

The price of the corresponding call option is therefore

$$c_{fut}(0) = 3.2890.$$

The put-call parity for futures options reads

$$p_{fut}(0) = e^{-rT}K - e^{-rT}F_0 + c_{fut}(0).$$

This can be obtained from the standard Black-Scholes put-call parity by substituting $s = e^{-rT}F_0$ everywhere. Using put-call parity we obtain

$$p_{fut}(0) = 6.2149.$$

4. (a) The zero coupon bond prices satisfy

$$p(0, T_i) = e^{-r(0,T_i)}K.$$

Here we have $T_1 = 1$ and this gives the zero rate

$$r(0, 1) = 3.0047\%.$$

Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + Kp(t, T_n)$$

For the two year coupon bond the coupon is $c^2 = 0.03 \cdot 1 \cdot 100 = 3$ and the formula reads

$$99.9063 = 3p(0,1) + (3 + 100)p(0,2).$$

Using that $p(0,1) = 0.9704$ we obtain that $p(0,2) = 0.9417$ and this results in the zero rate

$$r(0,2) = 3.0034\%$$

Finally, the coupon for the three year bond is $c^3 = 0.02 \cdot 1 \cdot 100 = 2$ and the formula reads

$$95.104 = 2p(0,1) + 2p(0,2) + (2 + 100)p(0,3).$$

Using that $p(0,1) = 0.9704$ and $p(0,2) = 0.9417$, we obtain that $p(0,3) = 0.8949$ and this results in the zero rate

$$r(0,3) = 3.7014\%.$$

(b) We have the following relationship between forward rates and spot rates

$$r(t,T)(T-t) = r(t,S)(S-t) + f(t;S,T)(T-S).$$

Thus

$$f(t;S,T)(T-S) = r(t,T)(T-t) - r(t,S)(S-t)$$

and we get the one year forward rate for the second year is

$$f(0;1,2)(2-1) = r(0,2) \cdot 2 - r(0,1) \cdot 1 = 3.0034 \cdot 2 - 3.0047 \cdot 1 = 3.0021\%,$$

and the one year forward rate for the third year is

$$f(0;2,3)(3-2) = r(0,3) \cdot 3 - r(0,2) \cdot 2 = 3.701443 \cdot 3 - 3.003426 \cdot 2 = 5.0975\%.$$

(c) The value of a forward rate agreement where you pay the rate $\bar{R}_s$ (quoted as a simple rate) over the interval $[S,T]$ on the principal K is given by

$$\Pi_0 = p(0,T)K\left[L(0;S,T) - \bar{R}_s\right](T-S).$$

For this exercise we have

$$\Pi_0 = p(0,3)10^6\left[L(0;2,3) - 0.05\right](3-2).$$

Here $L(0;2,3)$ denotes today's LIBOR rate for the third year. The LIBOR rate $L(0;S,T)$ is the same rate as the forward rate $f(0;S,T)$, only quoted as a simple rate. We thus have

$$1 + L(0;S,T)(T-S) = e^{f(0;S,T)(T-S)}$$

In this exercise we need the LIBOR rate for the third year and this is obtained from

$$1 + L(0;2,3) \cdot 1 = e^{f(0;2,3) \cdot 1},$$

so

$$L(0;2,3) = e^{f(0;2,3) \cdot 1} - 1 = e^{0.050975} - 1 \approx 0.05230$$

The value of the forward contract is thus

$$\Pi_0 = p(0,3)10^6\left[L(0;2,3) - 0.05\right](3-2) = 0.8949 \cdot 10^6(0.05230 - 0.05) \cdot 1 \approx 2055.$$

5. (a) If we note that

$$|S_T - K| = 2\max\{S_T - K, 0\} - S_T + K,$$

we can write down the price Π of the option as

$$\begin{aligned}\Pi_t &= e^{-r(T-t)}E^Q[2\max\{S_T - K, 0\} - S_T + K|\mathcal{F}_t] \\ &= 2c(t, S_t, K, T, r, \sigma) - S_t + e^{-r(T-t)}K.\end{aligned}$$

Here $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. Using the hints on the last page of the exam we obtain

$$\Pi_t = 2\left(S_t\Phi[d_1(t, S_t)] - e^{-r(T-t)}K\Phi[d_2(t, S_t)]\right) - S_t + e^{-r(T-t)}K.$$

where

$$\begin{aligned}d_1(t, s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t, s) &= d_1(t, s) - \sigma\sqrt{T-t}.\end{aligned}$$

(b) For the T-claim $X = \phi(S_T) = S_T^2$ we have

$$\begin{aligned}\Pi_0[X] &= e^{-rT}E^Q\left[S_T^2\right] \\ &= e^{-rT}\left(V(S_T) + \{E[S_T]\}^2\right).\end{aligned}$$

Recall that $S_T = S_0e^Z$ where $Z \in N\left((r - \sigma^2/2)T, \sigma\sqrt{T}\right)$. To simplify notation let

$$m = (r - \sigma^2/2)T \quad \text{and} \quad \Sigma = \sigma\sqrt{T}.$$

Now use the hint at the end of the exam concerning the expectation and variance of a lognormal random variable to obtain

$$\begin{aligned}\Pi_0[X] &= e^{-rT}\left(S_0^2(e^{\Sigma^2} - 1)e^{2m+\Sigma^2} + \left\{S_0e^{m+\Sigma^2/2}\right\}^2\right) \\ &= e^{-rT}\left(S_0^2(e^{\Sigma^2} - 1)e^{2m+\Sigma^2} + S_0^2e^{2m+\Sigma^2}\right) \\ &= e^{-rT}S_0^2e^{2m+\Sigma^2}\left((e^{\Sigma^2} - 1) + 1\right) \\ &= e^{-rT}S_0^2e^{2m+2\Sigma^2} \\ &= e^{-rT}S_0^2e^{2rT+\sigma^2T} = S_0^2e^{(r+\sigma^2)T}.\end{aligned}$$