KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 19 2015 08.00-13.00.

Answers and brief solutions.

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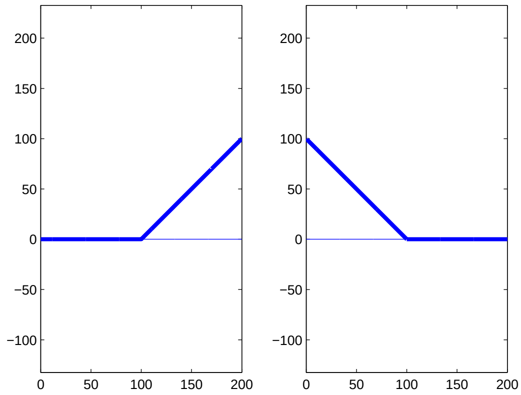
1. (a) This exercise can be solved in two ways.

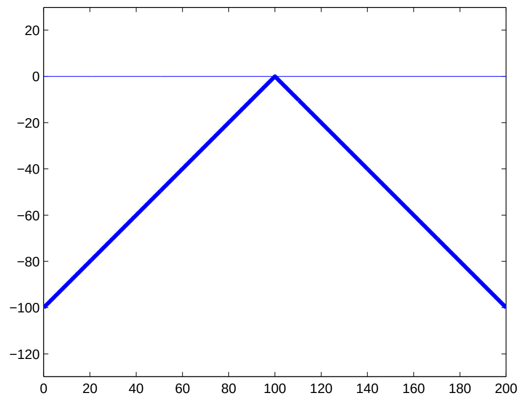
i. Risk-neutral valuation. The martingale measure should satisfy
$$e^{r\Delta t}S_t = qS^u_{t+\Delta t} + (1-q)S^d_{t+\Delta t},$$
so
$$q = \frac{e^{r\Delta t}S_t - S^d_{t+\Delta t}}{S^u_{t+\Delta t} - S^d_{t+\Delta t}}.$$
With numbers
$$q = \frac{e^{0.03\cdot 0.25}60 - 55}{65-55} \approx 0.545169.$$
The price of the option in three months is
$$\Pi(T) = X = \max\{S_T - K\} = \max\{S_T - 60\} = \begin{cases} 5 & \text{if } S_T = 65 \\ 0 & \text{if } S_T = 55 \end{cases}$$
The price at time $t = 0$ is then found as
$$\Pi(0) = E^Q\left[\frac{\Pi(T)}{B_T}\right] = e^{-rT}E^Q[X]$$
or with numbers
$$\Pi(0) = e^{-0.03\cdot 0.25}\{0.545169 \cdot 5 + (1 - 0.545169)\cdot 0\} \approx 2.71.$$

ii. Replicating portfolio. The number of stocks in the replicating portfolio is $y = \Delta$, i.e.
$$y = \Delta = \frac{\Delta\Pi}{\Delta S} = \frac{5-0}{65-55} = 0.5.$$
To find the amount of cash x you should have in the bank account solve
$$xe^{r\Delta t} + yS^u_{t+\Delta t} = \Pi^u(t+\Delta t)$$
with numbers
$$xe^{0.03\cdot 0.25} + 0.5\cdot 65 = 5.$$
This yields $x = -e^{-0.03\cdot 0.25}27.5$ and the value of the option at time $t = 0$ is equal to the value of the replicating portfolio at time $t = 0$, that is
$$x + yS_0 = -e^{-0.03\cdot 0.25}27.5 + 0.5\cdot 60 \approx 2.71.$$

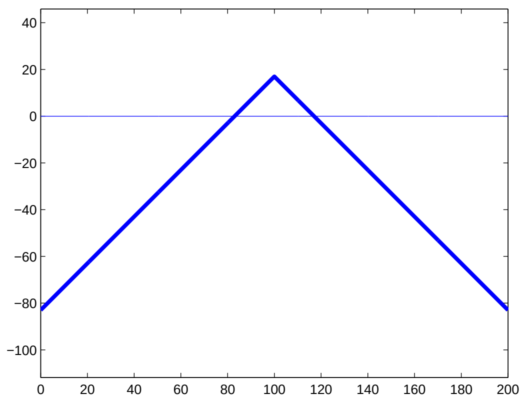
(b) The payoff from the call option and the put option look as follows



Since the options have been sold to create the top straddle, the payoff from the top straddle is given by



To obtain the **net** payoff add the price of the options, so 8+9=17. The net payoff of the top straddle is therefore



2. (a) Recall that an exchange rate works like an asset paying a continuous dividend yield equal to the foreign interest rate. In this case the Japanese market is the domestic market, and the US market is the foreign market, so the continuous dividend yield should be set equal to the to the US interest rate. We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/2 = 1/4 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0513 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.0.9512 \end{aligned}$$

and the tree for the stock price is therefore

		138.1464
	131.4089	
125.0000		125.0000
	118.9037	
		113.1047

The continuous dividend yield δ, should be equal to the US interest rate, i.e. $\delta = 0.02$, whereas the rate r should be set to the Japanese interest rate, i.e. $r = 0.04$. Now the option price tree can be computed using

$$q = \frac{e^{(r-\delta)\Delta t} - d}{u - d} \approx 0.5376.$$

Note that the dividend yield comes in to play when computing the martingale probabilities. For pricing we also need the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0101}$$

and the result is

		0.0000
	2.2890	
6.2981		5.0000
	11.0963	
		16.8953

In each node, that is not an end node, the value is obtained as

$$\max\{130 - S_t, \frac{1}{1.0101}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_t is the current stock price, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 11.0963. The price of the option is thus 6.2981.

(b) If we denote the option price by Π we have that

$$\Delta = \frac{\partial \Pi}{\partial s} \approx \frac{\Delta \Pi}{\Delta s}.$$

This gives us

$$\Delta = \frac{2.2890 - 11.0963}{131.4089 - 118.9037} \approx -0.7043.$$

3. (a) i. If we denote by $C(t, S_t, K, T)$ the price at time t of a European call option with strike price K and exercise date T written on the stock, and by $P(t, S_t, K, T)$ the price at time t of a European put option with the same strike price and exercise date as the call, and also having the stock as underlying, then according to put-call parity we have

$$P(t, S_t, , K, T) = Ke^{-r(T-t)} + C(t, S_t, K, T) - S_t.$$

ii. If we denote the price of a derivative written on the underlying stock by Π we have by definition that

$$\Delta = \frac{\partial \Pi}{\partial s}.$$

For a European call option in the standard Black-Scholes framework this yields

$$\Delta_{call} = \Phi[d_1(t, s)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$d_1(t, s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}.$$

Using the definition of delta and the expression for delta for a European call option we obtain

$$\Delta_{put} = \frac{\partial\left[Ke^{-r(T-t)} + C(t, s, K, T) - s\right]}{\partial s} = \Phi[d_1(t, s)] - 1,$$

or using the properties of the cumulative distribution function of the standard normal distribution

$$\Delta_{put} = -\Phi[-d_1(t, s)].$$

(b) i. We have that the forward price is given by

$$f(0, T) = \frac{\Pi_0(S_T)}{p(0, T)} = e^{rT} E^Q\left[\frac{S_T}{B_T}\right].$$

This forward price will make the initial value of the forward contract zero. To compute the expectation note that we have the formula

$$S_0 = E^Q\left[\frac{S_T}{B_T} + \sum_{t_i \leq T} \frac{\Delta D_{t_i}}{B_{t_i}}\right]$$

Here we can insert any T and therefore we have that

$$S_0 = E^Q\left[\frac{S_{0.5}}{B_{0.5}} + \frac{\Delta D_{3/12}}{B_{3/12}}\right], \tag{1}$$

but also

$$S_0 = E^Q\left[\frac{S_{3/12-}}{B_{3/12-}}\right]. \tag{2}$$

If we use that $\Delta D_{3/12} = 0.05S_{3/12-}$ and insert this and (2) into (1), we obtain

$$S_0 = E^Q\left[\frac{S_{0.5}}{B_{0.5}} + \frac{0.05S_{3/12-}}{B_{3/12-}}\right] = E^Q\left[\frac{S_{0.5}}{B_{0.5}}\right] + 0.05S_0$$

or

$$E^Q\left[\frac{S_{0.5}}{B_{0.5}}\right] = (1-0.05)S_0$$

The forward price is therefore

$$f(0,0.5) = e^{0.02\cdot 0.5}(1-0.05)100 \approx 95.95.$$

ii. Since we have just computed the forward price and interest rates are deterministic, which means that forward prices and futures prices are equal, we can use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)}F_t$) with parameters

$$F_0 = 95.95, \quad K = 95, \quad \sigma = 0.20, \quad r = 0.02, \quad T = 0.5$$

or the Black-Scholes formula with parameters

$$s = (1-\delta)S_0 = 95, \quad K = 95, \quad \sigma = 0.20, \quad r = 0.02, \quad T = 0.5.$$

The price of the call option is therefore

$$c(0) = 5.8146.$$

4. (a) The zero coupon bond prices satisfy

$$p(0,T_i) = e^{-r(0,T_i)T_i}K.$$

Here we have $T_1 = 0.5$, $K = 100$, and $r(0,0.5) = 1.5\%$ and this yields the bond price

$$p(0,0.5) = 100e^{-0.015\cdot 0.5} = 99.25.$$

For $T_2 = 1$ $K = 100$, and $r(0,0.5) =$ we obtain the bond price

$$p(0,1) = 100e^{-0.02\cdot 1} = 98.02.$$

Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t,T_i) + Kp(t,T_n)$$

For the two year coupon bond the coupon is $c^2 = 0.03 \cdot 1 \cdot 100 = 3$ and the formula reads

$$p_{fixed}(0) = 3p(0,1) + (3+100)p(0,2).$$

Using that $p(0,1) = 0.980199$ and that $p(0,2) = e^{-0.025\cdot 2} = 0.951229$ this results in the coupon bond price

$$p_{fixed}(0) = 100.92.$$

(b) The swap rate is set so as to make the value of the fixed and the floating leg equal, i.e.

$$cp(0,1) + (c+K)p(0,2) = K.$$

Using that $c = R_s \cdot 1 \cdot K$, we obtain

$$R_s = \frac{1 - p(0.2)}{p(0,1) + p(0,2)} = \frac{1 - 0.951229}{0.980199 + 0.951229} \approx 0.02525$$

The swap rate is thus $R_s = 2.53\%$.

(c) The value of a forward rate agreement where you pay the rate $\bar{R}_s$ (quoted as a simple rate) over the interval $[S,T]$ on the principal K is given by

$$\Pi_0 = p(0,T)K\left[L(0;S,T) - \bar{R}_s\right](T-S).$$

For this exercise we have

$$\Pi_0 = p(0,2)10^6\left[L(0;1,2) - 0.035\right](2-1).$$

Here $L(0;1,2)$ denotes today's LIBOR rate for the second year. The LIBOR rate $L(0;S,T)$ is the same rate as the forward rate $f(0;S,T)$, only quoted as a simple rate. We thus have

$$1 + L(0;S,T)(T-S) = e^{f(0;S,T)(T-S)}$$

We have the following relationship between forward rates and spot rates

$$r(t,T)(T-t) = r(t,S)(S-t) + f(t;S,T)(T-S).$$

Thus

$$f(t;S,T)(T-S) = r(t,T)(T-t) - r(t,S)(S-t)$$

and we get the one year forward rate for the second year is

$$f(0;1,2)(2-1) = r(0,2)\cdot 2 - r(0,1)\cdot 1 = 2.5\cdot 2 - 2.0\cdot 1 = 3.0\%,$$

In this exercise we need the LIBOR rate for the second year and this is obtained from

$$1 + L(0;1,2)\cdot 1 = e^{f(0;1,2)\cdot 1},$$

so

$$L(0;1,2) = e^{f(0;1,2)\cdot 1} - 1 = e^{0.03} - 1 \approx 0.030455$$

The value of the forward contract is thus

$$\Pi_0 = p(0,2)10^6\left[L(0;1,2) - 0.035\right](2-1) = 0.951229\cdot 10^6(0.030455 - 0.035)\cdot 1 \approx -4324.$$

5. (a) Denote the payoff function by ϕ and note that

$$\phi(S_T) = -a + \max\{S_T - x_1, 0\} - 2\max\{S_T - x_2, 0\} + \max\{S_T - x_3, 0\}.$$

The price of the contract is therefore

$$\begin{aligned}\Pi_t &= e^{-r(T-t)}E^Q[-a+\max\{S_T-x_1,0\}-2\max\{S_T-x_2,0\}\\ &\quad +\max\{S_T-x_3,0\}|\mathcal{F}_t]\\ &= -e^{-r(T-t)}a+c(t,S_t,x_1,T,r,\sigma)-2c(t,S_t,x_2,T,r,\sigma)\\ &\quad +c(t,S_t,x_3,T,r,\sigma)\\ &= -e^{-r(T-t)}a+c(t,S_t,0.95S_t,T,r,\sigma)-2c(t,S_t,S_t,T,r,\sigma)\\ &\quad +c(t,S_t,1.05S_t,T,r,\sigma).\end{aligned}$$

Here $c(t,s,K,T,r,\sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. If we want the price of the claim to be zero a should be chosen as

$$a = e^{r(T-t)}\left[c(t,S_t,0.95S_t,T,r,\sigma)-2c(t,S_t,S_t,T,r,\sigma)+c(t,S_t,1.05S_t,T,r,\sigma)\right],$$

where $c(t,s,K,T,r,\sigma)$ is given by the Black-Scholes formula. Inserting the following parameters

$$t=0, \quad S_0=100, \quad \sigma=0.20, \quad r=0.02, \quad T=0.25,$$

we obtain

$$a = 0.9774.$$

(b) Since the interest rate is zero the option price is given by the following formula

$$C_{Bach}(0) = E^Q[\max\{S_T-K,0\}] = E^Q[(S_T-K)I_{\{S_T-K\geq 0\}}].$$

Since

$$S_T-K = S_0+\sigma S_0 V_T - K,$$

we see that $S_T-K \in N(S_0-K,\sigma^2 S_0^2 T)$. Now apply the stated result with $\mu = S_0-K$, "$\sigma^2 = \sigma^2 S_0^2 T$", $l=0$, and $h=\infty$, and note that $1-\Phi(x)=\Phi(-x)$ and $\phi(x)=\phi(-x)$.