KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 17 2016 08.00-13.00.

Answers and brief solutions.

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1. (a) Replicating portfolio. The price of the option in three months is

$$\Pi(T) = X = \max\{S_T - K\} = \max\{S_T - 97\} = \begin{cases} 18 & \text{if } S_T = 115 \\ 0 & \text{if } S_T = 75 \end{cases}$$

The number of stocks in the replicating portfolio is $y = \Delta$, i.e.

$$y = \Delta = \frac{\Delta \Pi}{\Delta S} = \frac{18 - 0}{115 - 75} = 0.45.$$

To find the amount of cash x you should have in the bank account solve

$$xe^{r\Delta t} + yS^u_{t+\Delta t} = \Pi^u(t + \Delta t)$$

with numbers

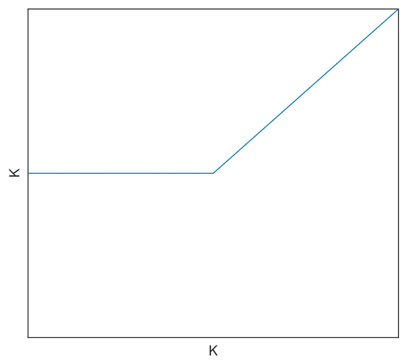
$$xe^{0.05 \cdot 0.25} + 0.45 \cdot 115 = 18.$$

This yields $x = -e^{-0.05 \cdot 0.25} 33.75 \approx 33.33$ and the replicating portfolio at time $t = 0$ is therefore

$$(x, y) = (33.33, 0.45),$$

meaning that you should have \$33.33 deposited in the bank account and own 0.45 shares of the stock.

(b) i. The payoff function of the protective put is found in the picture below



ii. If we denote by $C(t, K)$ the price at time t of a European call option with strike price K and exercise date T written on the stock, and by $P(t, K)$ the price at time t of a European put option with the same strike price and exercise date as the call, and also having the stock as underlying, then according to put-call parity we have

$$P(t, K) = e^{-r(T-t)}K + C(t, K) - S(t).$$

The arbitrage price of the protective put is therefore

$$\Pi_{prot_put}(t) = S(t) + P(t, K) = e^{-r(T-t)}K + C(t, K).$$

Using the Black-Scholes formula with the parameters with parameters

$$S_0 = 90, \quad K = 85, \quad \sigma = 0.30, \quad r = 0.02, \quad T = 0.25.$$

we find that

$$\Pi_{prot_put}(t) = e^{-r(T-t)}K + C(t, K) = e^{-0.02*0.25}85 + 8.3713 = 92.9473.$$

2. (a) We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/2 = 1/4 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.1912 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.8395 \end{aligned}$$

Now we make a tree for S^* starting at

$$S_0^* = S_0 - PV_0(div) = 100 - e^{-0.03 \cdot 3/12}5 = 95.0374.$$

The tree for S^* is therefore

$$\begin{array}{ccc} & & 134.8644 \\ & 113.2129 & \\ 95.0374 & & 95.0374 \\ & 79.7798 & \\ & & 66.9717 \end{array}$$

Now the stock price tree can be computed using $S_t = S_t^* + PV_t(div)$. The stock price tree becomes

$$\begin{array}{ccc} & & 134.8644 \\ & 113.2129 & \\ 100.0000 & & 95.0374 \\ & 79.7798 & \\ & & 66.9717 \end{array}$$

Now the option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.4778,$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0075}$$

and the result is

$$\begin{array}{ccc} & & 34.8644 \\ & 18.2129 & \\ 8.6364 & & 0.0000 \\ & 0.0000 & \\ & & 0.0000 \end{array}$$

In each node the value is obtained as

$$\max\{S_{t-} - 100, \frac{1}{1.00}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_{t-} is the current stock price **just before dividend payment** (so at time $t = 0.25$ you should use the value in the stock price tree + \$5), and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 18.2129. The price of the option is thus 8.6364.

(b) The price of a call option will increase if there are no dividends, since the option is more likely to be in the money if the stock pays no dividends (the stock price will be higher).

3. (a) i. We have that the futures price (which is equal to the forward price, since interest rates are deterministic) is given by

$$F(0,T) = \frac{\Pi_0(S_T)}{p(0,T)} = e^{rT} E^Q \left[\frac{S_T}{B_T} \right].$$

This futures price will make the initial value of the futures contract zero. To compute the expectation note that we have the formula

$$S_0 = E^Q \left[\frac{S_T}{B_T} + \sum_{t_i \le T} \frac{\Delta D_{t_i}}{B_{t_i}} \right]$$

so

$$S_0 = E^Q \left[\frac{S_{0.5}}{B_{0.5}} + \frac{\Delta D_{3/12}}{B_{3/12}} \right]. \tag{1}$$

If we use that $\Delta D_{3/12} = 5$ we obtain

$$S_0 = E^Q \left[\frac{S_{0.5}}{B_{0.5}} \right] + \frac{5}{B_{3/12-}} = E^Q \left[\frac{S_{0.5}}{B_{0.5}} \right] + e^{-0.02*0.25}5$$

or

$$E^Q \left[\frac{S_{0.5}}{B_{0.5}} \right] = S_0 - e^{-0.02*0.25}5$$

The futures price is therefore

$$F(0,0.5) = e^{0.02 \cdot 0.5}(100 - e^{-0.02*0.25}5) \approx 95.98.$$

ii. Use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)}F_t$) with parameters

$$F_0 = 95.98, \quad K = 96, \quad \sigma = 0.30, \quad r = 0.02, \quad T = 0.25.$$

The price of the call option on the futures price is therefore

$$c_{fut}(0) = 5.7002.$$

(b) Let $P(t, s)$ denote the price of the portfolio at time t when $S_t = s$. We then have that

$$P(t, s) = c(t, s) - p(t, s).$$

Here $c(t, s)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s. The price of the corresponding put option is denoted by $p(t, s)$. By definition we have that

$$\Delta_P(t, s) = \frac{\partial P(t, s)}{\partial s}.$$

The computation of this derivative will simplify considerably if if we use put-call-parity, that is

$$p(t, s) = Ke^{-r(T-t)} + c(t, s) - s.$$

We then have that

$$P(t, s) = c(t, s) - \Big(Ke^{-r(T-t)} + c(t, s) - s\Big) = s - Ke^{-r(T-t)}.$$

The delta of the portfolio is thus

$$\Delta_P(t, s) = \frac{\partial P(t, s)}{\partial s} = 1.$$

4. (a) The zero coupon bond prices satisfy

$$p(0, T_i) = e^{-r(0,T_i)}K.$$

Here we have $T_1 = 0.5$ and this gives the zero rate

$$r(0, 0.5) = 1.0025\%.$$

Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + Kp(t, T_n).$$

For the 18 month coupon bond trading at 103.693 the coupon is $c = 0.03 \cdot 1 \cdot 100 = 3$ paid annually, and the formula reads

$$103.693 = 3p(0, 0.5) + (3 + 100)p(0, 1.5).$$

Using that $p(0, 0.5) = 0.995$ we obtain that $p(0, 1.5) = 100.708/103$ and this results in the zero rate

$$r(0, 1.5) = 1.5002\%.$$

For the 18 month coupon bond trading at 100.7326 the coupon is $c = 0.02 \cdot 0.5 \cdot 100 = 1$ paid semi-annually, and the formula reads

$$100.7326 = 1p(0, 0.5) + 1p(0, 1) + (1 + 100)p(0, 1.5).$$

Using that $p(0, 0.5) = 0.995$, and that $p(0, 1.5) = 100.708/103$ we obtain that $p(0, 1) = 0.985095$ and this results in the zero rate

$$r(0, 1) = 1.5017\%.$$

Finally, the coupon for the two year bond is $c = 0.02 \cdot 1 \cdot 100 = 2$ and the formula reads

$$99.9707 = 2p(0,1) + (2+100)p(0,2).$$

Using that $p(0,1) = 0.985095$ we obtain that $p(0,2) = 0.960789$ and this results in the zero rate

$$r(0,3) = 2.0000\%.$$

(b) We have the following relationship between forward rates and spot rates

$$r(t,T)(T-t) = r(t,S)(S-t) + f(t;S,T)(T-S).$$

Thus

$$f(t;S,T)(T-S) = r(t,T)(T-t) - r(t,S)(S-t)$$

and we get that the six month forward rate beginning in one year is

$$f(0;1,2)(2-1) = r(0,2)\cdot 2 - r(0,1)\cdot 1 = 2.0000 \cdot 2 - 1.5017 \cdot 1 = 2.4983,$$

so $f(0;1,2) = 2.4983\%$.

(c) The value of a forward rate agreement where you pay the rate $\bar{R}_s$ (quoted as a simple rate) over the interval $[S,T]$ on the principal K is given by

$$\Pi_0 = p(0,T)K\left[L(0;S,T) - \bar{R}_s\right](T-S).$$

For this exercise we have

$$\Pi_0 = p(0,2)10^6\left[L(0;1,2) - 0.017\right](2-1).$$

Here $L(0;1,2)$ denotes today's LIBOR rate for the one-year period starting in one year. The LIBOR rate $L(0;S,T)$ is the same rate as the forward rate $f(0;S,T)$, only quoted as a simple rate. We thus have

$$1 + L(0;S,T)(T-S) = e^{f(0;S,T)(T-S)}$$

In this exercise we need the LIBOR rate for the one-year period starting in one year and this is obtained from

$$1 + L(0;1,2)\cdot 1 = e^{f(0;1,2)\cdot 1},$$

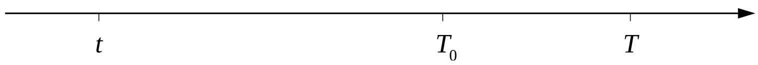
so

$$L(0;1,2) = e^{f(0;1,2)\cdot 1} - 1 = e^{0.024983} - 1 \approx 0.0252978.$$

The value of the forward contract is thus

$$\Pi_0 = p(0,2)10^6\left[L(0;1,2) -\right] = 0.960789 \cdot 10^6(0.0252978 - 0.026) \approx -675.$$

5. (a) The time line looks as follows



The claim can be replicated in the following way:

- At time t invest $e^{-r(T_0-t)}$ in the risk free asset (since this is the price of the derivative the replicating portfolio should cost this much). This means that you buy $e^{-r(T_0-t)}/B(t)$ risk free assets.
- At time T_0 use the money invested in the risk free asset to buy stocks. The amount of money which can be invested in stocks is given by
$$\frac{e^{-r(T_0-t)}}{B(t)}B(T_0) = 1,$$
where we have used that $B(u) = e^{ru}$. Thus you should buy $1/S(T_0)$ stocks.
- At time T your portfolio will be worth
$$\frac{1}{S(T_0)}S(T),$$
i.e. exactly as much as the claim itself, and since the portfolio strategy described is self-financing we have found a replicating portfolio.

(b) The price of the pay later option is given by
$$\Pi_t = e^{-r(T-t)}E^Q\left[(S_T - K - p)I\{S_T > K\}|\,\mathcal{F}_t\right]$$
where the super-index Q on the expectation indicates that the expectation should be taken under the martingale measure Q, and
$$I\{x > K\} = \begin{cases} K & \text{if } x > K, \\ 0 & \text{otherwise} \end{cases}$$
Rewriting this a bit we have
$$\Pi_t = e^{-r(T-t)}E^Q\left[(S_T - K)I\{S_T > K\}|\,\mathcal{F}_t\right] - e^{-r(T-t)}pE^Q\left[I\{S_T > K\}|\,\mathcal{F}_t\right]$$
Now, we can use Black-Scholes formula for the first term, since it is a plain vanilla call option. For the second term recall that $S_T = S_t e^Z$, where $Z \in N((r - \sigma^2/2)(T - t), \sigma^2(T - t))$. The expectation in the second term can now be computed and the result is
$$Q(S_T > K) = N[d_2(t)]$$
(for more details see the solution to exercise 5 b of the exam given in June 2014). The price of the pay later option at time t is thus
$$\Pi_t = S_tN[d_1(t)] - e^{-r(T-t)}KN[d_2(t)] - e^{-r(T-t)}pN[d_2(t)],$$
and the correct premium p is therefore given by
$$p = e^{rT}S_0\frac{N[d_1(0)]}{N[d_2(0)]} - K.$$